Math 304 **Midterm 1** **Due 11/2/2006**

This exam is due Thursday, November 2, in class. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

---

1. Brualdi, Problems 2.7, 3.7, 3.38, 4.16, 4.23, 4.29, 5.20, 5.30.

2. Give a combinatorial proof of the following identity: for any positive integer n,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

   Hint: Show that the left-hand side is the number of even subsets of $[n]$, while the right-hand side is the number of odd subsets of $[n]$. Then give a bijection between the set of even subsets of $[n]$ and the set of odd subsets of $[n]$. (Don't forget to show that your map is, in fact, a bijection.)

3. (extra) Construct a cyclic Gray code algorithm for the set of k-subsets of $[n]$ (or, alternatively, for the set of binary strings of length n containing exactly k 1s and $n-k$ 0s). In other words, the successor each k-subset $S \subseteq [n]$ should be obtained by deleting a single element in S and adjoining a single element not in S (i.e. the successor of S has exactly $k-1$ elements in common with S). (For strings, the successor is obtained by switching a single $(0,1)$-pair, not necessarily in adjacent positions.)

   (a) Describe your algorithm recursively, i.e. using the same algorithm on $[n-1]$.

   (b) Prove by induction on n that your recursive construction indeed works and that the resulting Gray code is indeed cyclic (i.e. the first subset and the last subset on the list have $k-1$ elements in common).

   (c) Find a nonrecursive way to obtain the successor of a given k-subset S of $[n]$ (i.e. you cannot generate the Gray code from the beginning through S, but must use only the properties of S to find its successor).

   (d) Prove that your nonrecursive construction does indeed yield the Gray code successor of each k-subset of $[n]$.