Math 304 **Homework 3** **Due 11/30/2006**

1. Brualdi, Chapter 8, Problems 8^{**}, 12^{***}, 19^{***}.

2. Brualdi, Chapter 7, Problems 11^{**}, $30c^{*}$, 32^{*}, 35^{**}, 37^{**}.

3. Prove that binomial coefficients satisfy the following identity:

$$\binom{m}{n} = (n+1)\binom{m+1}{n} - (m+2)\binom{m}{n-1}.$$

4. We saw in the homework that the series $\sum_{k=1}^{\infty} k^n x^k$ is obtained by applying n times the combined operation $x\frac{d}{dx}$ of "differentiating, then multiplying by x" to $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. As a result, we have

$$\sum_{k=1}^{\infty} k^n x^k = \frac{x}{(1-x)^{n+1}} \sum_{m=0}^{n-1} E(n,m)x^m$$

for some coefficients $E(n,m)$, $0 \le m \le n-1$. These coefficients $E(n,m)$ are called *Eulerian numbers.*

   (a) Prove that the Eulerian numbers satisfy the following recurrence relation: $E(1,0) = 1$, $E(1,m) = 0$ for $m \neq 0$, and

$$E(n,m) = (m+1)E(n-1,m) + (n-m)E(n-1,m-1), \quad n \ge 2, 0 \le m \le n-1.$$

   Hint: Apply $x\frac{d}{dx}$ one more time.

   (b) Use part (a) to prove that $E(n,m) = E(n,n-1-m)$. *Hint:* Show that $f(n,m) = E(n,n-1-m)$ satisfies the same recurrence relation as $E(n,m)$.

5. A *descent* of a permutation π of $[n]$ is a position $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. Let $A(n,m)$ be the number of permutations of $[n]$ with exactly m descents.

   (a) Prove that $A(n,m) = E(n,m)$. *Hint:* Prove that $A(n,m)$ satisfies the same recurrence relation as $E(n,m)$.

   (b) Give a combinatorial proof for Problem 4b. *Hint:* Read each permutation of $[n]$ from left to right, then from right to left.