Math 304 **Final Exam** **Due 12/13/2006**

This exam is due Wednesday, December 13, by 5pm, in my office. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. Brualdi, Problems 6.12, 6.17, 7.30(d).

2. Find the ordinary generating function for the *harmonic* numbers

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

   Hint: Recognize $\{H_n\}$ as the convolution of two simpler sequences and recall (or find) their generating functions.

3. Use the method of characteristic roots to solve the following recurrence relations:

   (a) $a_n = 3a_{n-1} - 2a_{n-2}$; $a_0 = 0$, $a_1 = 1$.

   (b) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$; $a_0 = 1$, $a_1 = 0$, $a_2 = 1$.

4. Find the coefficient of x^n in the power series of

$$f(x) = \frac{1}{(1-x^2)^2},$$

   first by the method of partial fractions, and second, give a much simpler derivation by using a substitution.

5. Find an explicit formula for $\{g_n\}_{n=0}^{\infty}$ given by

$$g_n = -ng_{n-1} + \sum_k \binom{n}{k} g_k g_{n-k}, \quad n \geq 2; \qquad g_0 = 0,\ g_1 = 1.$$

   Hint: You may want to use an exponential generating function and recall the exponential convolution. Alternatively, guess the exact formula for $f(n)$, then prove it by induction.

6. We want to find a formula for the nth derivative of $e^{e^x} = \exp(\exp(x))$. Differentiate it a few times, study the pattern, and conjecture the form of the answer in general, including some constants to be determined. Then find a recurrence formula for the constants in question, and identify them as some “famous” numbers that we have studied.

7. It is possible to extend the values $S(n,k)$ of the Stirling numbers of the second kind to negative integers n and k by induction, letting $S(0,0)=1$, $S(n,0)=0$ for $n \neq 0$, and

$$S(n-1,k-1) = S(n,k) - kS(n-1,k).$$

Find a simple closed-form expression for $S(n,k)$, where $n,k \leq 0$, in terms of known sequences with nonnegative arguments. *Hint:* Calculate $S(n,k)$ recursively for small negative n and k, compare the results with the sequences your learned in this course, then conjecture the general formula for $f(n,k) = S(-n,-k)$ $(n,k \geq 0)$, and prove it.

8. The *ballot numbers* $b(n,k)$ are defined as

$$b(n,k) = \frac{k}{2n+k}\binom{2n+k}{n}, \quad k \geq 1.$$

(a) Show that the nth Catalan number $C_n = b(n,1)$, and

$$b(n,k) = \frac{k}{n+k}\binom{2n+k-1}{n}, \quad b(n,k) = \binom{2n+k-1}{n} - \binom{2n+k-1}{n-1}.$$

(b) Consider an election with 2 candidates, R and D, where R got n votes and D got $n+k-1$ votes, $k \geq 1$. After each vote was cast (for R or for D), we calculated by how many votes D was currently ahead of R, and it turned out that D was never ahead of R by k votes or more (i.e. was behind R, tied with R, or ahead of R by $\leq k-1$ votes). In other words, the running total of votes for D was always less than the running total of votes for R plus k. Prove that the number of possible sequences of votes which satisfy this condition is $b(n,k)$. *Hint:* Use the second formula in part (a) and a combinatorial proof similar to that of $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

(c) *(extra credit)* Let $B_k(x) = \sum_{n=0}^{\infty} b(n,k)x^n$ and $C(x) = \sum_{n=0}^{\infty} C_n x^n$ be the ordinary generating functions for sequences $\{b(n,k)\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$, respectively. Prove that $B_k(x) = C(x)^k$. *Hint:* Start by looking at the last votes where the vote difference $D - R$ was $0, 1, 2, \ldots, k-2$ votes.