Math 552 **Homework 4** **Due 5/2/2005**

1. Give combinatorial proof of the q-binomial recursions

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^{k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

using the fact that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{m \geq 0} p(n-k, k, m) q^m,$$

where $p(n-k,k,m)$ is the number of unit-step south-east lattice paths from $(0,k)$ to $(n-k,0)$ with area m in the first quadrant under the path.

2. (a) Let $M = M(n; k, n-k) = \{1^k, 2^{n-k}\}$ be the multiset of n_1 1's, and n_2 2's. Let $S(M)$ be the set of all distinct permutations of M. Prove that

$$\sum_{\pi \in S(M)} q^{inv(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

where $inv(\pi)$ is the number of inversions of π. *Hint:* Use problem 1 and find an appropriate bijection.

(b) Let $M = M(n; n_1, \ldots, n_k) = \{1^{n_1}, \ldots, k^{n_k}\}$ be the multiset of n_1 copies of 1, n_2 copies of 2, etc., n_k copies of k. Let $n = n_1 + \cdots + n_k$. Let $S(M)$ be the set of all distinct permutations of M. Prove that

$$\sum_{\pi \in S(M)} q^{inv(\pi)} = \begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix}_q.$$

Hint: Use part 2(a) and induction on k.

3. For a permutation $\pi \in S_n$, let $Des(\pi) = \{i \mid \pi(i) > \pi(i+1)\}$ be the set of descents of π and define the *major index* of π as

$$maj(\pi) = \sum_{i \in Des(\pi)} i$$

Show that maj is Mahonian (i.e. has the same distribution as inv on S_n) by proving there is a bijection $\phi : S_n \to SE_n$ such that for any $\pi \in S_n$, $maj(\pi) = tot(\phi(\pi))$. (The n-tuple $\phi(\pi)$ is called the maj-code of π.) *Hint:* Given a permutation $\tau \in S_{n-1}$, insert n into τ in each possible position $i \in [n]$, to obtain permutations $\tau'_i \in S_n$, $i \in [n]$. Consider $maj(\tau'_i) - maj(\tau)$ for each $i \in Des(\pi)$ and each $i \notin Des(\pi)$. Formulate your hypothesis and prove it. Now, for any $\pi \in S_n$, construct maj-code of π by induction on n.