Math 552 **Homework 3** **Due 4/8/2005**

1. Use Lagrange inversion to find the following:

   (a) Let $C_n^{(\ell)}$ be the number of walks on or above the x-axis from $(0,0)$ to $((\ell+1)n, 0)$ with steps $(1,1)$ and $(1,-\ell)$, for some integer $\ell > 0$. Find $C_n^{(\ell)}$.

   (b) Let $C_{n,k}^{(\ell)}$ be the number of walks on or above the x-axis from $(0,0)$ to $((\ell+1)n+k, k)$ with steps $(1,1)$ and $(1,-\ell)$, for some integers $\ell > 0$, $n,k \geq 0$. Find $C_{n,k}^{(\ell)}$.

2. Find the general formula for
$$\frac{d^n}{dx^n}(e^{e^x})$$
the nth derivative of e^{e^x}. *Hint:* Differentiate e^{e^x} a few times, study the pattern, and conjecture the general form of the answer, including some constants to be determined. Then find a recurrence formula (recall that $(n+1)$-st derivative is the derivative of the nth derivative) and the initial values for the constants in question, and identify them as some "famous" numbers we have seen before.

3. (a) Let $P_1 = (a_1, b_1)$, $P_2 = (a_2, b_2)$, $Q_1 = (c_1, d_1)$, $Q_2 = (c_2, d_2)$ be integer lattice points in $\mathbb{Z}^2$. Suppose that $a_2 < a_1 < c_2 < c_1$ and $b_1 < b_2 < d_1 < d_2$. Consider pairs of unit-step northeast lattice lattice paths from $P = \{P_1, P_2\}$ to $Q = \{Q_1, Q_2\}$ (i.e. either $\{P_1 \to Q_1, P_2 \to Q_2\}$ or $\{P_1 \to Q_2, P_2 \to Q_1\}$). What can you say about the number of lattice path crossings in each of the two cases? Prove your claim.

   (b) Now let $P = \{P_1, \ldots, P_n\}$ to $Q = \{Q_1, \ldots, Q_n\}$, be sets of integer lattice points in $\mathbb{Z}^2$, where $P_i = (a_i, b_i)$, $Q_i = (c_i, d_i)$ for all $i \in [n]$, and
$$a_n < \cdots < a_1 < c_n < \cdots < c_1,$$
$$b_1 < \cdots < b_n < d_1 < \cdots < d_n.$$
   Let $\pi \in S_n$. Prove that the total number of crossings of unit-steps northeast lattice paths in the family
$$\{P_i \to Q_{\pi(i)} \mid 1 \leq i \leq n\}$$
   has the same parity as the number of inversions of π (i.e. number of pairs $i, j \in [n]$ such that $i < j$ but $\pi(i) > \pi(j)$).

   (c) Given the same families P and Q, let m_{ij} be the number of northeast lattice paths from P_i to Q_j. Find m_{ij}.

   (d) Let $M = [m_{ij}]_{1 \leq i,j \leq n}$, an $n \times n$ matrix. Prove that the number of noncrossing families of unit-step northeast lattice paths $P \to Q$ is equal to $\det M$.