Math 552 **Homework 2** **Due 2/28/2005**

1. Let $C_{n,k}$ be the number of Dyck walks from $(0,0)$ to $(2n,0)$ with unit steps $(1,1)$ and $(1,-1)$ that return to x-axis exactly k times (where $1 \leq k \leq n$). Find the generating function
$$C(t,u) = \sum_{n,k} C_{n,k} t^n u^k.$$

2. Let $f_{n,k}$ be the number of walks from $(0,0)$ to (n,k) with unit steps $(1,1)$ and $(1,-1)$ that stay on or above the x-axis. Find the generating function
$$F(t,u) = \sum_{n,k} f(n,k) t^n u^k.$$

   Hint: The last step is up or down unless it is from the x-axis, from which it is always up.

3. A *labeled* tree is a tree where each vertex is assigned a distinct label. Let $\mathcal{T}$ be the class of rooted labeled trees. Prove that
$$\mathcal{T} = \mathcal{Z} \times \mathcal{U}(\mathcal{T}).$$

   Let $T(t)$ be the exponential generating function of $\mathcal{T}$ with respect to number of vertices. Find the functional equation satisfied by $T(t)$.

4. An entry $\pi(i)$ of a permutation π is called a *left-to-right* (resp. *right-to-left*) *minimum* (resp. *maximum*) if for all $j < i$ (resp. $j > i$) we have $\pi(j) > \pi(i)$ (resp. $\pi(j) < \pi(i)$). Prove that the number of permutations of $[n]$ with exactly k left-to-right minima is equal to $c(n,k)$. Show that the same is true for the other three cases.

5. Give a combinatorial proof (no generating functions) that for all integers x, we have
$$\sum_{k=0}^{n} c(n,k) x^k = (x+n-1)_n.$$

6. In any permutation π has an *descent* (resp. a *ascent*) at position i if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). A permutation is called *alternating* or *up-down* if its descents and ascents alternate. Let $\mathcal{A}$ be the class of alternating permutations and let $A(t)$ be the exponential generating function for $\mathcal{A}$. Prove that
$$A(t) = \sec t + \tan t.$$

   Note: Thus, the coefficients $a_n = [t^n]A(t)$ are called *secant numbers* when n is even and *tangent numbers* when n is odd.