Math 552 **Homework 1** **Due 2/7/2005**

1. Use the formal power series identity $e^{tx}e^{ty} = e^{t(x+y)}$ in $\mathbb{C}[x,y][[t]]$ to derive the binomial formula
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. (a) Consider walks in the (x,y)-plane with unit steps $(1,0)$ east, $(-1,0)$ west, $(0,1)$ north, and $(0,-1)$ south. Prove that that the number of such n-step walks from $(0,0)$ to anywhere on the x-axis ($y=0$) that stay in the half-plane $y \geq 0$, is equal to C_{n+1}, the $(n+1)$-st Catalan number. *Hint:* Find the ordinary generating function (OGF) for the class of such walks using their recursive description, and notice how it is related to the OGF for C_n.

   (b) (extra) Give an explicit bijection between the set of n-step walks from the previous part and the set of Dyck paths of semilength $n+1$.

3. A *tree* is a connected graph with no cycles. A *rooted* tree is a tree with one distinguished vertex called the *root* (so a rooted tree must be nonempty). A *plane* tree (also called an *ordered* tree) is a tree embedded in the plane. Equivalently, the descendants of each vertex in a plane tree (the neighbors of our vertex that are one more edge away from the root) are arranged in a particular order from left to right. Different ordering yields a different plane tree. The size of a tree is the number of its vertices. Let $T(t)$ be the generating function for the class of rooted plane trees $\mathcal{T}$. Give two different recursive descriptions of $\mathcal{T}$ which correspond to functional equations:
$$T(t) = t + T(t)^2 \qquad \text{and} \qquad T(t) = \frac{t}{1 - T(t)}$$
and prove that the number of rooted plane trees on n vertices is C_{n-1}.

4. A *polyomino* is a finite connected union of square cells on the square lattice (the interior is connected). A polyomino P is *column-convex* if the intersection of P with any vertical line is connected. The size of a polyomino is the number of cells in it. Find the generating function for the number of column-convex polyominoes of size n.

   Hint: Use the example of directed column-convex polyominoes worked out in class.

5. The bottom cell in the leftmost column of a polyomino is called the *source*. A polyomino is *directed* if each cell can be reached from the source along a northeast path (i.e. only moving up or right). Give a recursive description of the class of directed column-convex (DCC) polyominoes that does not use marked-cell DCC polyominoes as auxiliary objects (as was done in class). Re-derive the generating function for the DCC polyominoes using your recursive description.

6. We know that the number of DCC polyominoes on n cells is F_{2n-2}, the $(2n-2)$-nd Fibonacci number, while the number of marked cell DCC polyominoes on n cells is F_{2n-1}. We also know that F_n is the number of compositions of n with parts of size 1 or 2. Give explicit bijections between:

   (a) DCC polyominoes on n cells and compositions of $2n-2$ with parts of size 1 or 2;

   (b) marked-cell DCC polyominoes on n cells and compositions of $2n-1$ with parts of size 1 or 2.