This exam is due Wednesday, December 14, by 5pm, in my office. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. Brualdi, Problems 6.12, 7.30(f).

2. Prove that binomial coefficients satisfy the following identity:

$$\binom{m}{n} = (n+1)\binom{m+1}{n} - (m+2)\binom{m}{n-1}.$$

3. We saw in the homework that the series $\sum_{k=1}^{\infty} k^n x^k$ is obtained by applying n times the combined operation xD of "differentiating, then multiplying by x" to $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. As the result, we have

$$\sum_{k=1}^{\infty} k^n x^k = \frac{x}{(1-x)^{n+1}} \sum_{m=0}^{n-1} E(n,m) x^m$$

for some coefficients $E(n,m)$, $0 \le m \le n-1$. These coefficients $E(n,m)$ are called *Eulerian numbers.*

   (a) Prove that the Eulerian numbers satisfy the following recurrence relation: $E(0,m) = 1$ for $m = 0$, $E(0,m) = 0$ for $m \neq 0$, and

$$E(n,m) = (m+1)E(n-1,m) + (n-m)E(n-1,m-1), \quad n \ge 1, 0 \le m \le n-1.$$

   Hint: Apply xD one more time.

   (b) Use part (a) to prove that $E(n,m) = E(n,n-1-m)$. *Hint:* Show that $f(n,m) = E(n,n-1-m)$ satisfies the same recurrence relation as $E(n,m)$.

4. (extra) A *descent* of a permutation π of $[n]$ is a position $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. Let $A(n,m)$ be the number of permutations of $[n]$ with exactly m descents.

   (a) Prove that $A(n,m) = E(n,m)$. *Hint:* Prove that $A(n,m)$ satisfies the same recurrence relation as $E(n,m)$.

   (b) Give a combinatorial proof for Problem 3b. *Hint:* Read each permutation of $[n]$ from left to right, then from right to left.

5. It is possible to extend the values $S(n, k)$ of the Stirling numbers of the second kind to negative integers n and k by induction, letting $S(0, 0) = 1$, $S(n, 0) = 0$ for $n \neq 0$, and

$$S(n-1, k-1) = S(n, k) - kS(n-1, k).$$

Find a simple closed-form expression for $S(n, k)$, where $n, k \leq 0$, in terms of known sequences with nonnegative arguments. *Hint:* Calculate $S(n, k)$ recursively for small n and k, compare the results with the sequences your learned in this course, then conjecture the general formula and prove it.

6. (extra) Let $h(n)$ be the number of circular words of length n over the alphabet $[k]$. Give a bijective proof that

$$nh(n) = \sum_{d|n} \phi\left(\frac{n}{d}\right) k^d,$$

where ϕ is the Euler's totient function.