

1. The *gamma function* is defined for any real  $\alpha > 0$  as follows:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$$

This integral converges for any positive real  $\alpha$  (see Problems 9.5.50-52).

- (a) Show that  $\Gamma(1) = 1$ .  
 (b) Use integration by parts to show that  $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ .  
 (c) Show that for  $\alpha = n$ , a positive integer, we have  $\Gamma(n + 1) = n!$ .  
 (d) Use part (b) to show how  $\Gamma(\alpha)$  can be extended for any real  $\alpha$  except negative integers.  
*Remark:* Thus, gamma function extends the factorial function to all real numbers except negative integers.  
 (e) Use substitution to show that

$$\Gamma(\alpha) = 2 \int_0^{\infty} x^{2\alpha-1} e^{-x^2} dx \quad \text{and} \quad \Gamma(\alpha) = \int_0^1 \left( \ln \frac{1}{u} \right)^{\alpha-1} du.$$

Use the formula on the left and Example 5, Section 9.3, to find  $\Gamma\left(\frac{1}{2}\right)$ .

2. The *beta function*  $B(\alpha, \beta)$  is defined for any  $\alpha, \beta > 0$  as follows:

$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du.$$

- (a) Show by a change of variables that  $B(\alpha, \beta) = B(\beta, \alpha)$ .  
 (b) Integrate by parts to show that

$$B(\alpha, \beta) = \frac{\alpha-1}{\beta} B(\alpha-1, \beta+1) = \frac{\beta-1}{\alpha} B(\alpha+1, \beta-1)$$

- (c) Assume that  $\alpha = m$ ,  $\beta = n$ , where  $m$  and  $n$  are positive integers. Use the result in part (b) repeatedly to show that

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

*Remark:* It can be proven (take Math 265) that the last equation actually holds for all  $\alpha, \beta$  for which  $\Gamma$  can be defined (see Problem 1(d)). Thus, beta function extends the binomial coefficients to all real values except negative integers.