

In the problems below, you may use the following theorem.

**Theorem 1** (*AM  $\geq$  GM*) For any positive integer  $k$  and any nonnegative real numbers  $x_1, x_2, \dots, x_k$ , their arithmetic mean is at least their geometric mean, i.e.

$$\frac{x_1 + x_2 + \dots + x_k}{k} \geq \sqrt[k]{x_1 x_2 \dots x_k}.$$

Moreover, the equality is achieved if and only if  $x_1 = x_2 = \dots = x_k$ . (In other words, we have an equality if and only if all  $x_i$ 's are equal, and we have a strict inequality if at least two  $x_i$ 's are different.)

You may not, however, use the facts you know about the number  $e$ .

1. (a) Let  $k = n + 1$  and  $x_1 = x_2 = \dots = x_n = 1 + \frac{1}{n}$  and  $x_{n+1} = 1$  in Theorem 1. Prove that the sequence  $\{a_n\}$ , where

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

is monotone increasing.

- (b) Let  $k = n + 1$  and  $x_1 = x_2 = \dots = x_n = 1 - \frac{1}{n}$  and  $x_{n+1} = 1$  in Theorem 1. Prove that the sequence  $\{b_n\}$ , where

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

is monotone decreasing. *Hint:* There is an additional step here compared to the previous part.

2. (a) Prove that for any  $n \geq 1$ , we have  $a_n < b_n$ .  
 (b) Prove that  $\{a_n\}$  has an upper bound and  $\{b_n\}$  has a lower bound.  
 (c) Prove that both  $\{a_n\}$  and  $\{b_n\}$  converge, and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

*Remark:* As you may remember, that limit is  $e$ .