Math 301 **Homework 3** **Due 11/12/04**

Part A.

1. Prove that $(a_1a_2\ldots a_n) = (a_1a_2)(a_2a_3)\ldots(a_{n-1}a_n)$.

2. Let V be the group of symmetries of a regular rectangle. Prove that V is isomorphic to a subgroup of A_4.

   Remark: After you solve this problem, you may also denote this subgroup of A_4 by V. V is called the Klein's 4-group. The notation V comes from German *vier*, which means "four."

3. Determine the number of cyclic permutations in S_n. *Hint:* Take any permutation $\pi \in S_n$ and write it in one-line notation. Now put parentheses around π to get a cyclic permutation $\pi' \in S_n$ in cycle notation. How many different permutations π yield the same cyclic permutation π' in this way? Now use your answer to solve the problem.

Part B.

1. Given a permutation $\pi \in S_n$, an *inversion* of π is a pair of positions (i, j) such that $i < j$ but $\pi(i) > \pi(j)$. (In one-line notation, it is any pair of entries of π where the larger entry is to the left of the smaller entry.) Define $inv(\pi)$ to be the number of inversions of π. Prove that π is even if and only if $inv(\pi)$ is even. (In other words, you can write π as a product of $inv(\pi)$ 2-cycles.) *Hint:* Try a proof by induction. For the induction step, multiply π by 2-cycles to move the letter n to the rightmost position. How many 2-cycles are used for this if $n = \pi(j)$ for some $1 \leq j \leq n$?

2. A subgroup N of G is called *normal* if $yN = Ny$ for any $y \in G$. If N is a normal subgroup of G, we write $N \lhd G$. Prove that $V \lhd A_4$. You may use the Cayley table for A_4 on p. 104 of the textbook.

3. Prove that the dihedral group D_4 and the group of quaternions Q (see Cayley table on p. 89) are not isomorphic. *Hint:* It's not enough to say that you can't map $\rho \mapsto a$, $\phi \mapsto b$, and have an isomorphism. You need to prove that *no mapping works*, this or any other. To do that, try a proof by contradiction. Assume $D_4 \cong Q$. What can you say then about the orders of elements in each group and the number of elements of each order? Now find the order of each element in D_4 and each element in Q, then the number of elements of each order in each group. Make a conclusion.

4. Prove that the set of elements of finite order in an abelian group forms a subgroup. (This subgroup is called the *torsion subgroup*.) Is the same thing true for non-abelian groups?

Part C.

1. Suppose G is a group that has exactly one nontrivial proper subgroup (i.e. not G itself and not $\{e\}$). Prove that G is cyclic and $|G| = p^2$ for some prime p.