Math 301 **Homework 2** **Due 10/04/04**

Part A.

1. Let a and b be nonzero integers. Prove that $d = \gcd(a, b)$ if and only if $a' = a/d$ and $b' = b/d$ are relatively prime integers.

2. (a) Use induction to prove that $0^2 + 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every integer $n \geq 0$.

   (b) Use induction to prove that $0^3 + 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for every integer $n \geq 0$.

Part B.

1. (Chinese Remainder Theorem) Let $r_1, r_2, m_1, m_2 \in \mathbb{Z}$ be such that $m_1, m_2 \neq 0$ and $\gcd(m_1, m_2) = 1$. Prove that $(\exists n \in \mathbb{Z})(n \equiv r_1 \pmod{m_1} \ \wedge \ n \equiv r_2 \pmod{m_2})$ (that is, $n = m_1 q_1 + r_1$ for some integer q_1 and $n = m_2 q_2 + r_2$ for some integer q_2). *Hint:* One way to solve this is to use the fact that $\gcd(m_1, m_2) = 1$ and Theorem 0.2 to find an integer n_1 such that $n_1 \equiv 1 \pmod{m_1}$ and $n_1 \equiv 0 \pmod{m_2}$, and an integer n_2 such that $n_2 \equiv 0 \pmod{m_1}$ and $n_2 \equiv 1 \pmod{m_2}$. Then use n_1 and n_2 to find n.

2. (a) List all the elements of the group $SL(2, \mathbb{Z}_2)$ and write down the *Cayley table* (also called the multiplication table) for this group. *Hint:* Some elements are not obvious, don't forget that $-1 \equiv 1 \pmod 2$.

   (b) Do the same for the group S_3 of bijections from the set $\{1, 2, 3\}$ to itself (the binary operation is the composition of functions).

3. (a) Check that $SL(n, \mathbb{Z})$ is a group under matrix multiplication, whereas $GL(n, \mathbb{Z})$ is not. *Hint:* Recall the formula for the inverse matrix using cofactors and the determinant.

   (b) Find matrices $x, y \in SL(2, \mathbb{Z})$ such that $x^4 = y^6 = I_2$ but $(xy)^n \neq I_2$ for any integer $n \neq 0$.

   Hint: Look for the simplest examples. Search among matrices with entries equal to $0, 1, -1$. If your product xy is one of the matrices I have in mind, you'll be able to use induction to find $(xy)^n$ for any $n \geq 0$. Then use inverse to find $(xy)^n$ for $n < 0$.

4. Let G be a set with a binary operation $*$ on G such that:

   (a) G is closed under $*$.

   (b) $*$ is associative.

   (c) $(\exists e \in G)(\forall x \in G)(e * x = x)$ (e is called a left identity of G).

   (d) $(\forall x \in G)(\exists l(x) \in G)(l(x) * x = e)$. ($l(x)$ is called a left inverse of x.)

   Prove that G is a group. (So you must prove that $x * l(x) = e$ and $x * e = x$ for e and all $x \in G$).

   Hint: Consider x, $l(x)$, $l(l(x))$ and some cleverly constructed products of those such as $l(l(x)) * l(x) * x * l(x)$ and $x * l(x) * x$. Do not assume $*$ is commutative! First prove a left inverse of an element is also a right inverse, and hence the two-sided inverse (as in a group). Then prove a left identity is also a right identity, and hence the two-sided identity (as in a group).

Part AA.

1. (a) Let G be a group. Suppose that $z^2 = e$ for every $z \in G$. Prove that G is abelian. *Hint:* Given $x, y \in G$, simplify $xyxyyx$ in two ways.

   (b) Let G be a group. Suppose that there is an integer $n \geq 0$ such that for every $x, y \in G$,

   $$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}, \quad (xy)^{n+2} = x^{n+2} y^{n+2}.$$

   Prove that G is abelian.

   Hint: Use cancellation and substitution laws several times. Note that $(xy)^{n+1} = (xy)^n xy$ and $(xy)^{n+2} = (xy)^{n+1} xy$. Do not assume the group operation is commutative!

2. Construct Cayley tables for groups $U(10)$ and $U(12)$. In what ways are they similar? In what ways are they different? *Hint:* Consider orders of the groups as well as products, orders and inverses of their elements.

Part C.

1. Give an example of a nonabelian group G such that for some integer $n > 0$,

   $$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1} \quad \text{for any } x, y \in G.$$

   Hint: We've seen that group before. Just find an n that works for all $x, y \in G$.

2. Choose your x, y as Problem B3b so that they also *generate* $SL(2, \mathbb{Z})$, i.e. so that every $z \in SL(2, \mathbb{Z})$ can be expressed as a product of a string of x's, y's and their inverses.

   Hint: Again, look for simplest examples. Search among matrices with entries equal to $0, 1, -1$. Then make use of the Euclidean algorithm. Try to perform the Euclidean algorithm on the top row of entries of a matrix in $SL(2, \mathbb{Z})$ using multiplication by your chosen matrices until you get the identity matrix I_2.