Part A.

1. In each part, prove that the two statements are not logically equivalent. To do that, find counterexamples P and Q which make one of the statements true and the other false. Does one of the statements in each pair imply the other? If so, which implies which? *Hint:* Easy counterexamples will do.

   (a) $(\forall x)(P(x) \vee Q(x))$ and $(\forall x)(P(x)) \vee (\forall x)(Q(x))$, where $P(x)$ and $Q(x)$ are some logical statements whose truth depends on the value of variable x.

   (b) $(\exists x)(P(x) \wedge Q(x))$ and $(\exists x)(P(x)) \wedge (\exists x)(Q(x))$, where $P(x)$ and $Q(x)$ are some logical statements whose truth depends on the value of variable x.

   (c) $(\forall x)(\exists y)(P(x,y))$ and $(\exists y)(\forall x)(P(x,y))$, where $P(x,y)$ is some logical statement whose truth depends on the value of variables x and y. *Hint:* The first statement reads: "for every x there is a y such that $P(x,y)$ is true," while the second statement reads: "there is a y such that for every x, $P(x,y)$ is true."

2. Let S be a set. Call S an *I-set* if it has a nonempty proper subset S_0 (i.e. $S_0 \subseteq S$, $S_0 \neq S$, $S_0 \neq \emptyset$) and there is a bijection (a 1-1 and onto function) $\varphi : S_0 \to S$. (So, $\mathbb{Z}$ and $\mathbb{N}$ are I-sets.) Call S an *F-set* set if S is not an I-set (in other words, there is no bijection between S and any of its nonempty proper subsets).

   Note: this is just an ad hoc definition rather than a standard term.

   (a) Prove that the empty set, $\emptyset$, is an F-set. *Hint:* Proof by contradiction, takes ≤ 30 seconds.

   (b) Let S be a set and $T \subseteq S$. Prove that if S is an F-set then T is an F-set. *Hint:* Prove the contrapositive: if T is an I-set then S is an I-set.

   Remark: You might have an intuition as to what I-sets and F-sets really are. However, here you can only use the definitions given in this problem.

3. Let S be a set, $a, b \in S$. Which of the following are reflexive, symmetric, transitive, and which are equivalence relations?

   (a) S is any set, $(a,b) \in R$ iff $a \neq b$.

   (b) S is the set of invertible matrices, $(a,b) \in R$ if $b = a^{-1}$.

   (c) S is the set of all polygons in the plane, $(a,b) \in R$ iff b is obtained from a by a rotation.

   (d) Same, but replace "rotation" with "translation."

   (e) m is a fixed nonnegative integer, $S = C^m(\mathbb{R})$ is the set of functions from $\mathbb{R}$ to $\mathbb{R}$ that have the mth derivative, $(a,b) \in R$ iff $a^{(m)} = b^{(m)}$ (i.e. a and b have the same mth derivative).

Part B.

1. Which of these are equivalence relations (see Problem A3), which are not, and why?

   (a) $S = \bigcup_{\alpha \in T} S_\alpha$ where all the sets S_α are nonempty and mutually disjoint (and index α ranges over some set T), $a \sim b$ iff both a and b are in the same S_α.

   (b) S is the collection of all sets, $a \sim b$ iff there is a bijection between set a and set b.

   (c) $S = C^\infty(\mathbb{R})$ is the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. functions that have all possible derivatives), $(a, b) \in R$ iff $a^{(n)} = b^{(n)}$ for some $n \geq 0$. *Note:* Here, as opposed to Problem A3e, n is not fixed but depends on a and b.

2. (a) What is wrong with the following proof that symmetry and transitivity imply reflexivity? "Let $(a, b) \in R$, then $(b, a) \in R$ by symmetry, so $(a, a) \in R$ by transitivity (using $c = a$)."

   (b) Suggest an alternative property close to reflexivity such that symmetry and transitivity do imply this alternative property.

Part BB.

1. Let S be a set such that for some element $e \in S$ there exists a bijection $\varphi : S \to S \setminus \{e\}$. Prove that there exists a injection from $\mathbb{N}$ to S.

   Hint: Consider elements $\{\varphi^n(e) \mid n \geq 0\} = \{e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots\}$, and prove they are all distinct, i.e. for $m, n \geq 0$, prove that $m \neq n$ implies $\varphi^m(e) \neq \varphi^n(e)$. In other words, prove that for any $n \geq 0$, $\varphi^{n+1}(e) \notin \{e, \varphi(e), \varphi^2(e), \ldots, \varphi^n(e)\}$.

   Remark: This proves that $\mathbb{N}$ is the smallest I-set (up to a bijection).

Part C.

1. For sets, use the definitions of F-sets and I-sets of Problem A2 above. Let R and S be F-sets. Prove that $R \cup S$ is an F-set.

   Remarks:

   (a) Results and proofs of Problems A2 and BB1 may be useful here.

   (b) Try a proof by contradiction. Say both R and S are F-sets, but $T = R \cup S$ is an I-set. Then there is a nonempty proper subset $T_0 \subseteq T$ such that there is a bijection $\varphi : T \to T_0$. Now $T_0 \subsetneq T$, so there is an element $e \in T \setminus T_0$. Consider elements $e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots$ in T. Each of them is either in R or in S. Now let $E = \{e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots\}$, $E_R = E \cap R$ and $E_S = E \cap S$. Prove that at least one of E_R or E_S must be an I-set, then prove that at least one of R or S must be an I-set. Now deduce the result of the problem.

(c) You may have an intuition already that I-sets are the infinite sets and F-sets are the finite sets, i.e. is those with, say, n elements for some integer $n \geq 0$. In that case, if R has n elements and S has m elements, then $R \cup S$ ought to have $\leq n+m$ elements. If you actually wish to use that intuition here, you need to establish by strict argument that your notion of a finite set (i.e. having n elements for some integer $n \geq 0$) is logically the same as the notion of an F-set from Problem A2. This is not so easy. Even if you do this, you are faced with the problem of $R \cup S$ and connection with addition in $\mathbb{N}$. At some point or points in your argument, some real ingenuity would be required.