Math 301 **Final Exam** **Due 12/16/04** **Name**____________________________

This exam is **due Thursday, December 16, in my office, Carver 456**. You may consult the text for this course, your notes taken in lecture, your homework, and sketches of solutions to homework problems from this or past semesters. Do not use other books or papers or materials from a library or consult with any person other than myself. Answers to all questions except multiple choice require a rigorous proof. Show your argument *neatly*. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

Part A.

This part consists of multiple-choice questions. Please *circle* the answer which is *always true* or *always correct*. Circle the *entire* answer, not just the letter labelling it. No work is required and no partial credit will be given for any part of this problem.

1. If G is a group of order 8740 and if H is a subgroup of G, then the following numbers are all possible as the number of cosets of H in G.

   (a) $\{76, 92, 95, 391\}$

   (b) $\{40, 76, 115, 475\}$

   (c) $\{5, 95, 101, 437\}$

   (d) $\{4, 76, 95, 1748\}$

   (e) $\{4, 95, 115, 2116\}$

2. Consider the group S_{27} of permutations of the set $\{1, 2, \ldots, 27\}$; so S_{27} is the symmetric group on 27 letters. Write H for the subgroup of all permutations $\sigma \in S_{27}$ such that $\sigma(9) = 9$ and $\sigma(18) = 18$. Then

   (a) $|S_{27} : H| = 702$

   (b) $|S_{27} : H| = 2$

   (c) $|S_{27} : H| = 25$

   (d) $|S_{27} : H| = 25!$

   (e) $|S_{27} : H| = 675$

3. Consider the subgroup G of $SL(2, \mathbb{R})$ consisting of all 2×2 upper triangular real matrices $\sigma = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ for which $a > 0$. Let G act on the set $\mathbb{R}^2$ according to the rule:

$$\sigma \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ a^{-1}y \end{bmatrix}$$

(In other words, the permutation of $\mathbb{R}^2$ associated with a matrix $\sigma \in G$ is the linear transformation that maps each point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\sigma\mathbf{v}$.) Let $\mathbf{v} \in \mathbb{R}^2$ be such that the stabilizer of $\mathbf{v}$ is the subgroup

$$\text{stab}(\mathbf{v}) = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid \text{all } b \in \mathbb{R} \right\}.$$

Then $\mathbf{v}$ can be:

(a) any point on the x-axis.
(b) any point on the x-axis except $(0, 0)$.
(c) any point on the x-axis except $(0, 0)$, $(1, 0)$, $(-1, 0)$.
(d) any point in the right half-plane, i.e. where $x > 0$.
(e) any point in the left half-plane, i.e. where $x < 0$.

4. Given the same G and its action on $\mathbb{R}^2$ as in the previous problem:

(a) There are no points $(x, y) \in \mathbb{R}^2$ whose stabilizer is the identity subgroup of G.
(b) There are points, but only finitely many, whose stabilizer is the identity subgroup of G.
(c) Except for the points on a certain straight line, all other points have stabilizer the identity subgroup of G.
(d) The points whose stabilizer is the identity subgroup of G form a straight line.
(e) None of the above is a true statement.

Part B.

In this part, partial credit will be given. Show all work here. All answers require rigorous proofs.

1. Let $\phi : G \to \overline{G}$ be a homomorphism with $\ker\phi = K$, and let H be a subgroup of G. Prove that the preimage of image of H under ϕ is HK, i.e. $\phi^{-1}(\phi(H)) = HK$.

2. Let G be a nonabelian group of order 21. It can be proved (and you may assume this) that G must have a unique subgroup H of order 7, which is normal in G.

   (a) Let $a, b \in G$ be such that $|a| = 3$ and $|b| = 7$ (they exist by Cauchy theorem, assume this). Determine the possible values of aba^{-1} (so that they are of the form $a^i b^j$ for some $0 \le i \le 2$ and some $0 \le j \le 6$).

   Hint: There are two answers. Determine which subset of G aba^{-1} should be in; that will determine i; then consider $aaaba^{-1}a^{-1}a^{-1}$; that will determine the two possible j's.

   (b) (extra) Prove that $G = \langle a, b \rangle$.

   (c) (extra) Prove that all nonabelian groups of order 21 are isomorphic to each other. (In other words, there is only one such group up to isomorphism.)

   Hint: Think about how the two answers in part (a) and $|H|$ are related. Also, given the answers in part (a), think about how aba^{-1} and $a^{-1}ba$ are related.

3. Give an example of a group G and subgroups $K \le H \le G$ such that $K \lhd H$ and $H \lhd G$ but $K \ntriangleleft G$.

4. If G is a group (not necessarily finite), $N \lhd G$ and $|G/N| = m$, prove that $x^m \in N$ for any $x \in G$. *Hint:* Consider the left coset $xN \in G/N$.

5. If G is nonabelian, prove that $Aut(G)$ is not cyclic. *Hint:* Prove the contrapositive. Recall that $Inn(G) \le Aut(G)$.

Part C.

1. Let $\phi : A_4 \to D_6$ be a homomorphism. Prove that $\ker\phi = A_4$ or $\ker\phi = V$. Determine the image $\phi(A_4)$ in each of the two cases. *Hint:* Start by determining the orders of elements in A_4 and possible orders of their images under ϕ.