Math 301 **Sketch of Solutions for Midterm 2** **04/26/04**

Part A.

Indicate whether each of the following statements is true or false. No work is required in this problem.

1. The statement "if a permutation σ is of order 2, then σ is a product of one or more disjoint 2-cycles"

   Answer: **(a)** is always true. If σ^2 is the identity permutation, so $\sigma(\sigma(i)) = i$ for any i. Thus, either $\sigma(i) = i$, or $\sigma(i) = j \neq i$ in which case $\sigma(j) = \sigma(\sigma(i)) = i$, hence element not fixed by σ is in some 2-cycle, so σ is a product of disjoint 2-cycles.

2. A product of a 2-cycle and a 3-cycle (not necessarily disjoint) is

   Answer: **(d)** neither a 3-cycle nor a 5-cycle. A 2-cycle is odd and a 3-cycle is even, so their product is odd. Any odd cycle is an even permutation, so if our product is a cycle, it must be an even cycle (which is an odd permutation).

3. Let G be a non-abelian finite group and let $\mathbb{Z}$ be the additive group of integers. Let $\phi : G \to \mathbb{Z}$ and $\psi : \mathbb{Z} \to G$ be two homomorphisms. Which of the following consists of *all* true statements? (*Remark:* A *zero* homomorphism is a homomorphism that maps every element to the identity.)

   Answer: **(a)** ϕ must be zero and ψ cannot be onto. Indeed, let $a \in G$, $a \neq e$. G is finite, so $|a|$ is also finite. Say $|a| = n$, then $a^n = e$. The map ϕ is a homomorphism, so $\phi(e) = 0$. But then $0 = \phi(e) = \phi(a^n) = n\phi(a)$ and $n = |a| > 0$, so $\phi(a) = 0$. Thus, $\phi(a) = 0$ for any $a \in G$, so ϕ is zero (so it cannot be 1-1 or onto). Also, $\mathbb{Z}$ is abelian and ψ is a homomorphism, so $\psi(\mathbb{Z}) \leq G$ also must be abelian. But G is non abelian, so $\psi(\mathbb{Z})$ cannot be the whole G, and hence ψ cannot be onto.

   Note also that ψ cannot be 1-1 either since $\mathbb{Z}$ is infinite and G is finite. On the other hand, ψ does not have to be zero. Note that $\langle e \rangle$ is abelian and G is not, so there is $a \in G$ such that $a \neq e$. Let $\psi : \mathbb{Z} \to G$ be a homomorphism such that $\psi(1) = a$ (so for any $n \in \mathbb{Z}$, $\psi(n) = a^n$). Then $\psi(\mathbb{Z}) = \langle a \rangle \neq \langle e \rangle$, so this ψ is not zero.

Part B.

1. Let $\phi : G \to H$ and $\psi : G \to K$ be two different homomorphisms. Suppose the *only* element $x \in G$ for which $\phi(x) = e_H$ and $\psi(x) = e_K$ is $x = e_G$. Prove: If $\phi(y) = e_H$ and $\psi(z) = e_K$, then $yz = zy$. *Hint:* Consider $yzy^{-1}z^{-1}$.

   We have

   $$\phi(yzy^{-1}z^{-1}) = \phi(y)\phi(z)\phi(y^{-1})\phi(z^{-1}) = e_H\phi(z)e_H^{-1}\phi(z)^{-1} = e_H,$$
   $$\psi(yzy^{-1}z^{-1}) = \psi(y)\psi(z)\psi(y^{-1})\psi(z^{-1}) = \psi(y)e_K\psi(y)^{-1}e_K^{-1} = e_K,$$

   therefore we must have $yzy^{-1}z^{-1} = e_G$, in other words $yz = zy$.

2. Let G be an abelian group of order 15, and let $a, b \in G$ be such that $|a| = 3$ and $|b| = 5$. Prove that G must be a cyclic group. *Hint:* Consider ab.

   What is $|ab|$? We know that $|ab|$ divides $|G| = 15$, so $|ab|$ is one of 1, 3, 5, 15.

   If $|ab| = 1$, then $ab = e$, i.e. $b = a^{-1}$. But then $5 = |b| = |a^{-1}| = |a| = 3$, which is impossible. Hence, $|ab| \neq 1$.

   If $|ab| = 3$, then $(ab)^3 = e$. But G is abelian, so $e = (ab)^3 = a^3b^3 = eb^3 = b^3$, so $b^3 = e$ and $b^5 = e$, so $b = be = bb^5 = b^6 = (b^3)^2 = e^2 = e$ and hence $|b| = 1 \neq 5$, which is impossible.

   If $|ab| = 5$, then $(ab)^5 = e$. But G is abelian, so $e = (ab)^5 = a^5b^5 = a^5e = a^5$, so $a^3 = e$ and $a^5 = e$, so $a = ae = aa^5 = a^6 = (a^3)^2 = e^2 = e$ and hence $|a| = 1 \neq 3$, which is impossible.

   Thus, the only possibility is $|ab| = 15 = |G|$, so $G = \langle ab \rangle$, and thus G is cyclic.

Remark: Actually, the assumptions that G is abelian and there are $a, b \in G$ such that $|a| = 3$ and $|b| = 5$ are redundant. With a bit more theory, it turns out that just the fact that $|G| = 15$ implies that G has elements a and b as above (Cauchy theorem) and that $ab = ba$ so that G is abelian, and therefore, as we have shown, cyclic.

3. Let G be a group and let $\alpha : G \to G$ be such that $\alpha(g) = g^{-1}$ for any $g \in G$. Prove that $\alpha \in \mathrm{Aut}(G)$ if and only if G is abelian.

   Since $\alpha(\alpha(g)) = (g^{-1})^{-1} = g$ for any $g \in G$, it follows that there is $\alpha^{-1} = \alpha$, so α is a bijection on G. Thus, we only need to prove that α is a homomorphism if and only if G is abelian.

   ($\Leftarrow$) If G is abelian, then $\alpha(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \alpha(g)\alpha(h)$ for any $g, h \in G$, so α is a homomorphism.

   ($\Rightarrow$) If α is a homomorphism, then $(gh)^{-1} = \alpha(gh) = \alpha(g)\alpha(h) = g^{-1}h^{-1} = (hg)^{-1}$ and hence $gh = hg$ for any $g, h \in G$. In other words, G is abelian.

4. Let G be a *finite* group. Suppose there exists $\phi \in \mathrm{Aut}(G)$ having no fixed points except e, i.e. $\phi(x) = x \implies x = e$ (and $x \neq e \implies \phi(x) \neq x$).

   (a) Prove that $(\forall x \in G)(\exists y_x \in G)(\phi(y_x) = y_x x)$. *Hint:* Prove that the function $f : G \to G$ such that $f(y) = y^{-1}\phi(y)$ is 1-1, then prove that f is, in fact, a bijection.

   Suppose $f(y) = f(z)$ for some $y, z \in G$, then $y^{-1}\phi(y) = z^{-1}\phi(z)$, so $zy^{-1} = \phi(z)\phi(y)^{-1} = \phi(zy^{-1})$. Hence, zy^{-1} is a fixed point of ϕ, so $zy^{-1} = e$, i.e. $z = y$. Therefore, f is 1-1.

   Since f is 1-1, we have $|f(G)| = |G|$ (thinking of G as the domain on both sides of the equation). But G is also the target set and G is *finite*, so f must be a bijection, hence, onto.

   Thus, for every $x \in G$ there exists a unique $y_x \in G$ such that $x = f(y) = y_x^{-1}\phi(y_x)$, i.e. such that $y_x x = \phi(y_x)$.

   (b) If, in addition, $\phi^2 = id_G$, prove that G must be abelian. *Hint:* Use part (a) and, for any $x \in G$, consider $\phi(\phi(y_x))$. Then use Problem B3.

   Now we also have that $\phi(\phi(g)) = g$ for any $g \in G$. Let $x \in G$, and let $y_x \in G$ be such that $y_x x = \phi(y_x)$. Then $y_x = \phi(\phi(y_x)) = \phi(y_x x) = \phi(y_x)\phi(x) = y_x x \phi(x)$, so $x\phi(x) = e$ for any $x \in G$. Thus, $\phi(x) = x^{-1}$ for any $x \in G$. Let $a, b \in G$, then $\phi(ab) = \phi(a)\phi(b)$ is equivalent to $(ab)^{-1} = a^{-1}b^{-1} = (ba)^{-1}$, i.e. $ab = ba$. Thus, any two elements of G commute, so G is abelian.