Math 301 | **Midterm 2** | **Due 04/16/04**

This exam is **due Friday, April 16, in class**. You may consult the text for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. Answers to problems in part B require rigorous proof. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

Part A.

This part consists of multiple-choice questions. Please *circle* the answer which is *always correct*. Circle the *entire* answer, not just the letter labelling it. No work is required and no partial credit will be given for any part of this problem.

1. The statement "if a permutation σ is of order 2, then σ is a product of one or more disjoint 2-cycles"
   (a) is always true.
   (b) is always false.
   (c) is sometimes true and sometimes false.

2. A product of a 2-cycle and a 3-cycle (not necessarily disjoint) is
   (a) neither a 2-cycle nor a 3-cycle.
   (b) neither a 2-cycle nor a 4-cycle.
   (c) neither a 3-cycle nor a 4-cycle.
   (d) neither a 3-cycle nor a 5-cycle.

3. Let G be a non-abelian finite group and let $\mathbb{Z}$ be the additive group of integers. Let $\phi : G \to \mathbb{Z}$ and $\psi : \mathbb{Z} \to G$ be two homomorphisms. Which of the following consists of *all* true statements? (*Remark:* A *zero* homomorphism is a homomorphism that maps every element to the identity.)
   (a) ϕ must be zero and ψ cannot be onto.
   (b) ϕ cannot be 1-1 and ψ may be onto.
   (c) ϕ may be 1-1 and ψ may be onto.
   (d) ϕ may be onto and ψ may be 1-1.
   (e) ϕ cannot be 1-1 and ψ must be zero.

Part B.

In this part, partial credit will be given, but all work must be shown and all answers require rigorous proofs.

1. Let $\phi : G \to H$ and $\psi : G \to K$ be two different homomorphisms. Suppose the *only* element of $x \in G$ for which $\phi(x) = e_H$ and $\psi(x) = e_K$ is $x = e_G$. Prove: If $\phi(y) = e_H$ and $\psi(z) = e_K$, then $yz = zy$. *Hint:* Consider $yzy^{-1}z^{-1}$.

2. Let G be an abelian group of order 15, and let $a, b \in G$ be such that $|a| = 3$ and $|b| = 5$. Prove that G must be a cyclic group. *Hint:* Consider ab.

3. Let G be a group and let $\alpha : G \to G$ be such that $\alpha(g) = g^{-1}$ for any $g \in G$. Prove that $\alpha \in \mathrm{Aut}(G)$ if and only if G is abelian.

4. Let G be a *finite* group. Suppose there exists $\phi \in \mathrm{Aut}(G)$ having no fixed points except e, i.e. $\phi(x) = x \implies x = e$ (and $x \neq e \implies \phi(x) \neq x$).

   (a) Prove that $(\forall x \in G)(\exists y_x \in G)(\phi(y_x) = y_x x)$. *Hint:* Prove that the function $f : G \to G$ such that $f(y) = y^{-1}\phi(y)$ is 1-1, then prove that f is, in fact, a bijection.

   (b) If, in addition, $\phi^2 = id_G$, prove that G must be abelian. *Hint:* Use part (a) and, for any $x \in G$, consider $\phi(\phi(y_x))$. Then use Problem B3.