This exam is **due Friday, March 26, in class**. You may consult the text for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. Answers to problems in part B require rigorous proof. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

Part A.

Indicate whether each of the following statements is true or false. No work is required for these problems and no partial credit will be given.

1. Since each permutation has an inverse, the product of all permutations in S_n is the identity permutation.

2. Upper-triangular real matrices with all 1's on the diagonal form a subgroup of $SL(n, \mathbb{R})$.

3. $(\mathbb{Z}, *)$, where $a * b = ab + a + b$, is a group.

4. Let G be a group and H a subgroup of G. Define the normalizer $N(H)$ of H by $N(H) = \{g \in G \mid g^{-1}Hg = H\}$. Then $N(H) = \bigcap_{h \in H} C(h)$, where $C(h)$ is the centralizer of an element h.

5. Not all groups of order 4 are cyclic.

Part B.

In this part, partial credit will be given, but all work must be shown and all answers require rigorous proofs.

1. Here are some entries of a Cayley table for a group G of order 6.

	u	v	w	x	y	z
u						z
v		w			x	
w			v			
x		y		u		
y						
z						

   Determine the rest of the entries of the above Cayley table. What known groups are isomorphic to G?

2. Let V be a group of symmetries of a rectangle. Prove that V is a subgroup of D_4. What are the elements of V? What is the order of V? Write the Cayley table for V. What is the order of each element of V? Is V abelian? Is V cyclic?

3. (a) Let G be a group and let $\rho, \phi \in G$. Prove by induction that $\rho\phi = \phi\rho^{-1}$ implies $\rho^n\phi = \phi\rho^{-n}$ for any integer n. *Hint:* Use induction on n.
   (b) Determine $C(\phi)$ in $D_4 = \{e, \rho, \rho^2, \rho^3, \phi, \phi\rho, \phi\rho^2, \phi\rho^3 \mid \phi^2 = \rho^4 = e,\ \rho\phi = \phi\rho^{-1}\}$.
   (c) Determine $C(\phi)$ in $D_5 = \{e, \rho, \rho^2, \rho^3, \rho^4, \phi, \phi\rho, \phi\rho^2, \phi\rho^3, \phi\rho^4 \mid \phi^2 = \rho^5 = e,\ \rho\phi = \phi\rho^{-1}\}$.
   (d) (extra) Determine $C(\phi)$ in $D_n = \{\phi^i\rho^j \mid i = 0, 1,\ j = 0, 1, \dots, n-1,\ \phi^2 = \rho^n = e,\ \rho\phi = \phi\rho^{-1}\}$. *Hint:* The answer depends on whether n is even or odd.

4. Let G be a group generated by two elements σ and τ. (In other words, G is the group that consists of all possible products of any number of factors each of which is one of σ, τ, σ^{-1}, τ^{-1}.)

   Remark: In each part of this problem, you may assume the results of the preceding parts.

   (a) If α is an element of G, explain carefully how you can conclude that α lies in the center $Z(G)$ of G if you know that $\alpha\sigma = \sigma\alpha$ and $\alpha\tau = \tau\alpha$. *Hint:* Use induction on the number of factors in a product of σ, τ, σ^{-1}, τ^{-1}.
   (b) Now suppose that G is generated by σ and τ, and that $\sigma^5 = e$, $\tau^3 = e$, and $\sigma^{-1}\tau = \tau\sigma$. Prove that $\tau^2 \in Z(G)$. *Hint:* Use part (a).
   (c) Now prove that $\tau \in Z(G)$. *Hint:* Use part (b).
   (d) From your results above and the fact that $\sigma^{-1}\tau = \tau\sigma$, conclude that $\sigma^2 = e$. *Hint:* Use part (c).
   (e) Finally, prove that $\sigma = e$, so G is actually generated by τ alone, and G is in fact cyclic of order 3. *Hint:* Use part (d).