

Part A.

- If G is a group and H is a subgroup of G , let $N(H) = \{a \in G \mid a^{-1}Ha = H\}$. $N(H)$ is called the *normalizer* of H . (Recall that $N(H)$ we showed in class that $N(H)$ is a subgroup of G .
 - Prove that $H \triangleleft N(H)$.
 - Prove that if K is a subgroup of G such that $H \triangleleft K$, then $K \leq N(H)$ (so $N(H)$ is the largest subgroup of G in which H is normal).
 - Consider the action of G on the set X of subgroups of G by conjugation (i.e. if $g \in G$ and $H \leq G$, then $g \cdot H = gHg^{-1}$). Let $H \leq G$. Prove that $\text{stab}(H) = N(H)$ under this action. Use the orbit-stabilizer theorem to prove that the number of subgroups of G conjugate to H is $|G : N(H)|$.

Part B.

- Prove that S_n is generated by two permutations, (12) and $(123 \dots n-1 n)$.
Hint: Use Problem B1 of HW5. Prove that each 2-cycle $(i \ i+1)$, $1 \leq i \leq n-1$ is generated by (12) and $(123 \dots n-1 n)$. Start by considering $(1 \ 2 \ 3 \dots n-1 \ n)(1 \ 2)(1 \ 2 \ 3 \dots n-1 \ n)^{-1}$ and go from there.
- Let G be a group. For any $g, h \in G$, define $[g, h] = ghg^{-1}h^{-1} = gh(hg)^{-1}$, the *commutator* of g and h . Let $\Delta(G)$ denote the subgroup generated by all commutators of elements of G , called the *commutator* subgroup of G , i.e.

$$\Delta(G) = \langle \{[g, h] \mid g, h \in G\} \rangle.$$

- Prove that $\Delta(G) \triangleleft G$. *Hint:* First prove that for any generator $[h, k] \in \Delta(G)$ and any $g \in G$, we have $g[h, k]g^{-1} = [ghg^{-1}, gkg^{-1}]$.
- Prove that $G/\Delta(G)$ is an abelian group. *Hint:* First prove that for any $g, h \in G$, gh and hg are in the same coset of $\Delta(G)$.

Part C.

- Consider the group G consisting of all 2×2 upper triangular real matrices $\sigma \in SL(2, \mathbb{R})$ with positive entries on the diagonal, i.e. $\sigma = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for which $\det(\sigma) = 1$ and $a > 0$, $c > 0$. Let G act on the the xy -plane (i.e. on the set $X = \mathbb{R}^2$) according to the rule:

$$\sigma \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cy \end{bmatrix}$$

Into how many orbits does the action of G partition the xy plane? What are they?

- Let G be a group, and let $K \leq H \leq G$ (so $K \leq G$ as well). Suppose that both $|G : H|$ and $|H : K|$ are finite (the groups G , H and K , however, may be infinite). Prove that

$$|G : K| = |G : H||H : K|.$$

If you cannot do the general case, prove the same result for a finite group G .