Math 301 **Homework 5** **Due 04/09/04**

Part A.

1. Prove that $\mathbb{Z}$ is isomorphic to a subgroup of $SL(2,\mathbb{Z})$. *Hint:* Find a generator for your subgroup. We've seen it in earlier homework.

2. Are all permutations of order 6 in S_6 necessarily cyclic? Prove your answer.

3. Let $\alpha, \beta \in S_n$. Prove that α and $\beta\alpha\beta^{-1}$ are both even or both odd.

4. Let G be a group of permutations on a set X (for example, S_n is a group of permutations on a set $X = [n] = \{1, 2, \ldots, n\}$).

   (a) Let $x \in X$, and define $stab(x) = \{\pi \in G \mid \pi(x) = x\}$. In other words, $stab(x)$ consists of all elements of G that leave x fixed. We call $stab(x)$ the *stabilizer* of x. Prove that $stab(x)$ is a subgroup of G for any $x \in X$. *Hint:* Just use the subgroup test.

   (b) Define a relation $\sim$ on X as follows: for any $x, y \in X$

   $$x \sim y \quad \text{if and only if} \quad x = \pi(y) \text{ for some } \pi \in G.$$

   Prove that $\sim$ is an equivalence relation. *Hint:* Check $\sim$ is reflexive, symmetric and transitive.

   Remark: The equivalence class of x under relation $\sim$ is called the *orbit* of x, denoted $orb(x)$. Thus, the action of G partitions X into disjoint orbits.

Part B.

1. Prove that S_n is generated by the $n-1$ transpositions of consecutive elements, i.e. by the set $\{(i\ i+1) \mid 1 \le i \le n-1\} = \{(12), (23), (34), \ldots, (n-1\ n)\}$.

   Hint: Show that it's enough to prove that every transposition (ij) is a product of transpositions of consecutive elements, e.g. $(13) = (12)(23)(12)$. It may help to consider the product $(i\ i+1)(i+1\ j)(i\ i+1)$.

2. Prove that D_{12} and S_4 are not isomorphic (even though they have the same number of elements).

3. Prove that A_n is generated by 3-cycles, i.e. every permutation in A_n can be expressed as a product of 3-cycles. *Hint:* First express a permutation in A_n as a product of transpositions (2-cycles). Now use the fact that $(ab)(ac) = (acb)$ and $(ab)(cd) = (abc)(adc)$ for distinct elements a, b, c, d.

Part C.

1. How many cyclic permutations does S_n have? Prove your answer. *Hint:* Write each cyclic permutation so that the cycle ends on n.

2. For which n, do all permutations of order n in S_n have to be cyclic? *Hint:* Recall that any positive integer is a product of powers of primes, which are relatively prime to each other.