Math 301 **Homework 4** **Due 03/12/04**

Part A.

1. Prove that $SL(2,\mathbb{Z}_2) \cong S_3$ (or that $SL(2,\mathbb{Z}_2) \cong D_3$).
2. Let H and K be subgroups of a group G. Prove that $H \cap K$ is also a subgroup of G.
3. Let G be a group, H be a subgroup of G.
   (a) For a given $x \in G$, let $x^{-1}Hx = \{x^{-1}hx \mid h \in H\}$. Prove that $x^{-1}Hx$ is a subgroup of G for any $x \in G$.
   Remark: K is called a subgroup *conjugate* to H if $K = x^{-1}Hx$ for some $x \in G$.
   (b) Let
   $$N = \bigcap_{x\in G} x^{-1}Hx.$$
   Prove that N is a subgroup of G such that $y^{-1}Ny = N$ for any $y \in G$.
   Hint: $y^{-1}Ny = N$ does *not* mean that $y^{-1}gy = g$ for each $g \in N$. It means only that $y^{-1}gy \in N$ for every $g \in N$ (i.e. $y^{-1}Ny \subseteq N$) *and* any $g \in N$ can be written as $g = y^{-1}hy$ for some $h \in N$ (i.e. $N \subseteq y^{-1}Ny$).
   Remark: A subgroup N of G is called *normal* if $y^{-1}Ny = N$ for any $y \in G$. If N is a normal subgroup of G, we write $N \lhd G$.

Part B.

1. Let $GL(2,\mathbb{C})$ be the group of 2×2 invertible complex matrices. Let
   $$\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad i^2 = -1.$$
   Let D be the smallest subgroup of $GL(2,\mathbb{C})$ containing α and β. Let H be the smallest subgroup of $GL(2,\mathbb{C})$ containing α and γ.
   (a) Prove that H has exactly 8 elements $\{\pm I, \pm x, \pm y, \pm z\}$ which satisfy
   $$x^2 = y^2 = z^2 = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad xy = z, \quad yz = x, \quad zx = y.$$
   Remark: H is called the *group of quaternions*.
   (b) Prove that D has exactly 8 elements and is isomorphic to the group D_4 of symmetries of a square.
   (c) *(extra)* Prove that D and H are not isomorphic.
   Hint: Find a property of D (such as number of elements, orders of elements, number of elements of a given order, etc.) which any group isomorphism $\phi : D \to H$ would preserve but which H does not have.
2. Let H and K be subgroups of a group G. Let $HK = \{hk \mid h \in H, k \in K\}$. Prove that HK is a subgroup of G if and only if $HK = KH$.

   Hint: For the "if" part, use the two-step subgroup test. For the "only if" part, prove that for any $h \in H$ and $k \in K$, we have $kh \in HK$ and $hk \in KH$ (think about inverses and their multiplication).

   Remark: Note that, in general, we need not have $HK = KH$ since $KH = \{kh \mid h \in H, k \in K\}$ and G is not necessarily abelian. Also note $HK = KH$ does not mean $hk = kh$ for each $h \in H$ and each $k \in K$. It only means that for any $h \in H$ and $k \in K$ there are some $h' \in H$ and $k' \in K$ such that $hk = k'h'$ (this says $HK \subseteq KH$) *and* for any $h \in H$ and $k \in K$ there are some $h'' \in H$ and $k'' \in K$ such that $kh = h''k''$ (this says $KH \subseteq HK$).