Math 301 **Homework 3** **Due 02/27/04**

Part A.

1. (a) List all the elements of the group $SL(2, \mathbb{Z}_2)$ and write down the *Cayley table* (in other words, the multiplication table) for this group.
   (b) Do the same for the groups D_3 and D_4 of symmetries of the regular triangle and the square, respectively.
   (c) Do the same for the group S_3 of bijections from the set $\{1, 2, 3\}$ to itself (the binary operation is the composition of functions).

2. (a) Let G be a group. Suppose that $x^2 = e$ for every $x \in G$. Prove that G is abelian. *Hint:* Simplify $xyxyyx$ in two ways.
   (b) Let G be a group. Suppose that there is an integer $n \geq 0$ such that for every $x, y \in G$,

$$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}, \quad (xy)^{n+2} = x^{n+2} y^{n+2}.$$

   Prove that G is abelian.

   Hint: Use cancellation and substitution several times. Note that $(xy)^{n+1} = (xy)^n xy$ and $(xy)^{n+2} = (xy)^{n+1} xy$. Do not assume the group operation is commutative!

Part B.

1. Let G be a set with a binary operation $*$ on G such that:
   (a) G is closed under $*$.
   (b) $*$ is associative.
   (c) There is an element $e \in G$ such that $e * x = x$ for all $x \in G$. (e is called a left identity of G).
   (d) For any $x \in G$, there is an element $l(x) \in G$ (which depends on x) such that $l(x) * x = e$. ($l(x)$ is called a left inverse of x.)

   Prove that G is a group. (So you must prove that $x * l(x) = e$ and $x * e = x$ for e and all $x \in G$).

   Hint: Consider x, $l(x)$, $l(l(x))$ and some cleverly constructed products of those such as $l(l(x)) * l(x) * x * l(x)$ and $x * l(x) * x$. Do not assume $*$ is commutative! First prove a left inverse of an element is also a right inverse, and hence the two-sided inverse (as in a group). Then prove a left identity is also a right identity, and hence the two-sided identity (as in a group).

2. Let $SL(2,\mathbb{Z})$ denote the set of 2×2 integer matrices whose determinant is 1.

   (a) Check that $SL(2,\mathbb{Z})$ is a group under matrix multiplication, whereas $GL(2,\mathbb{Z})$ (the set of all invertible integer matrices) is not.

   (b) Find matrices $x, y \in SL(2,\mathbb{Z})$ such that $x^4 = y^6 = I_2$ but $(xy)^n \neq I_2$ for any integer $n \neq 0$.

   Hint: Look for simplest examples. Search among matrices with entries equal to $0, 1, -1$.

Part C.

1. Give an example of a nonabelian group G such that for some integer $n > 0$,

$$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}$$

   for any $x, y \in G$.

2. Choose your x, y as Problem 2(b) so that they also *generate* $SL(2,\mathbb{Z})$, i.e. so that every $z \in SL(2,\mathbb{Z})$ can be expressed as a product of a string of x's, y's and their inverses.

   Hint: Again, look for simplest examples. Search among matrices with entries equal to $0, 1, -1$. Then make use of the Euclidean algorithm. Try to perform the Euclidean algorithm on the entries of a matrix in $SL(2,\mathbb{Z})$ using multiplication by your chosen matrices until you get the identity matrix I_2.