**Math 301** **Sketch of Solutions to Assignment 1** **02/07/04**

**Part A.**

1. (a)

$$(\forall x)(P(x)) \vee (\forall x)(Q(x)) \implies (\forall x)(P(x) \vee Q(x)),$$

but

$$(\forall x)(P(x) \vee Q(x)) \not\Longrightarrow (\forall x)(P(x)) \vee (\forall x)(Q(x)).$$

Indeed, to see that the second implication is false, it is enough to let $Q(x) = \neg P(x)$ (i.e. $Q(x)$ is "not $P(x)$"), for example $P(x) =$ "$x$ is even" and $Q(x) =$ "$x$ is odd". Then every integer is either odd or even, but not every integer is even and not every integer is odd.

(b)

$$(\exists x)(P(x) \wedge Q(x)) \implies (\exists x)(P(x)) \wedge (\exists x)(Q(x)),$$

but

$$(\exists x)(P(x)) \wedge (\exists x)(Q(x)) \not\Longrightarrow (\exists x)(P(x) \wedge Q(x)).$$

Again, to see that the second implication is false, let $Q(x) = \neg P(x)$, say $P(x) =$ "$x$ is even" and $Q(x) =$ "$x$ is odd". Then some integers are odd and some integers are even, but no integer is both odd and even.

(c)

$$(\exists y)(\forall x)(P(x,y)) \implies (\forall x)(\exists y)(P(x,y)),$$

but

$$(\forall x)(\exists y)(P(x,y)) \not\Longrightarrow (\exists y)(\forall x)(P(x,y))$$

To see that the second implication is false, let $P(x,y) =$ "$x = y$". Then for every $x$ we can find some $y$ equal to $x$, say, $x$ itself (note that different $y$'s work for different $x$'s), but there is no single $y$ that is simultaneously equal to every $x$ (note that no $y$ works for all $x$'s at the same time).

2. (a) The empty set, $\emptyset$, has no nonempty proper subset (since it has no nonempty subset), hence no bijection to a nonempty proper subset. Thus, the empty set is finite.

(b) We will prove the contrapositive: "If $S$ is a set and $T \subseteq S$, then $S$ is infinite if $T$ is infinite."

Indeed, suppose $T$ is infinite. Then there is a nonempty subset $T_0 \subseteq T$ such that $T_0$ is in 1-1 correspondence with $T$. Let $\varphi : T \to T_0$ be a bijection from $T$ to $T_0$. Define a function $f$ on $S$ as follows:

$$f(s) = \begin{cases} \varphi(s) & \text{if } s \in T, \\ s & \text{if } s \in S - T. \end{cases}$$

In other words, $f$ is $\varphi$ on $T$ and the identity function on $S - T$. Let $S_0 = (S - T) \cup T_0$. Then $S_0 = f(S)$ and $f$ is a bijection from $S$ to $S_0$, since

$$f^{-1}(s) = \begin{cases} \varphi^{-1}(s) & \text{if } s \in T, \\ s & \text{if } s \in S - T. \end{cases}$$

is also a function. It is also easy to show that $S - S_0 = T - T_0 \neq \emptyset$. Thus, there is a bijection from $S$ to its nonempty proper subset, so $S$ is infinite.

3. The answers are as follows:

| Part | Reflexive | Symmetric | Transitive | **Equivalence** |
|---|---|---|---|---|
| a | Y | Y | Y | Y |
| b | N | Y | N | N |
| c | Y | N | Y | N |
| d | Y | Y | Y | Y |
| e | Y | Y | Y | Y |
| f | N | Y | N | N |

**Part B.**

1. The answers are as follows:

| Part | Reflexive | Symmetric | Transitive | **Equivalence** |
|---|---|---|---|---|
| a | Y | Y | Y | Y |
| b | Y | Y | Y | Y |

2. (a) The "proof" implicitly assumes that we know there is an element $b$ (same as or different from $a$) such that $a \sim b$. However, the reflexivity property assumes no such thing, it simply states that $a \sim a$ for all $a$ with no additional assumptions. In math logic notation, we end up proving $(\forall a \in S)((\exists b \in S)(a \sim b) \implies a \sim a)$, not the unconditional $(\forall a \in S)(a \sim a)$.

   (b) We should simply make our implicit assumption explicit, so the new property should be: "For any element $a \in S$, if there exists an element $b \in S$ such that $a \sim b$, then $a \sim a$." In other words, if $a$ is equivalent to any element, then it is equivalent to itself.

3. Let $S$ be the smallest set such that there is an element $e \in S$ and there is a bijection $\varphi : S \to S - \{e\}$. ("Smallest" here means that if $T$ is another set with the same property and there is an injection from $T$ to $S$, then there is a bijection between $T$ and $S$. This corresponds to our intuition on finite sets, but covers both finite and infinite sets.)

   Consider a set $E = \{a_i \,|\, i \in \mathbb{N}\}$ defined as follows: $a_1 = e$, $a_{n+1} = \varphi(a_n)$ for any $n \geq 1$ (in other words, $a_n = \varphi^{n-1}(e)$). We claim that $a_i \neq a_j$ if $i \neq j$. Indeed, without loss of generality (WLOG), suppose that $i < j$. $i - j$ is a positive integer, so $i - j \geq 1$, so $\varphi^{i-j}(e) = \varphi(\varphi^{i-j-1}(e)) \in \varphi(S) = S - \{e\}$, hence $\varphi^{i-j}(e) \neq e$. $\varphi$ is a bijection, so $\varphi^{j-1}$ is a also a bijection as a composition of $j - 1$ bijections $\varphi$. Thus, $\varphi^{i-j}(e) \neq e$ implies that $\varphi^{j-1}(\varphi^{i-j}(e)) \neq \varphi^{j-1}(e)$, i.e. $\varphi^{i-1}(e) \neq \varphi^{j-1}\{e\}$, that is $a_i \neq a_j$.

   It is easy to see that $f : E \to \mathbb{N}$ given by $f(a_n) = n$ is a bijection. Hence, $E \simeq \mathbb{N}$. On the other hand, $E$ is a subset of $S$ and the restriction of $\varphi$ to $E$ is a bijection from $E$ to $E - \{e\}$. However, $S$ is the smallest set with such property, so there exists a bijection $g : S \to E$. But then $f \circ g : S \to \mathbb{N}$ is also a bijection, so $S \simeq \mathbb{N}$.

4. Problems A2 and B3 will both be useful here, so read their solutions first.

   Our proof is by contradiction. Assume $R$ and $S$ are finite, but $T = R \cup S$ is infinite. Then there is a bijection $\varphi$ from $T$ to some nonempty proper subset $T_0$ of $T$. $T_0 \subsetneq T$, so $T - T_0 \neq \emptyset$. Let $e \in T - T_0$. Construct a subset $E$ of $T$ as in Problem B2, in other words, $E = \{a_n \,|\, a_n = \varphi^{n-1}(e),\ n \in \mathbb{N}\}$. Then $E$ is infinite since $\varphi(E) = E - \{e\}$.

   Consider two subsets of $E$: $E_R = E \cap R$ and $E_S = E \cap S$. Then $E_R \subseteq R$, $E_S \subseteq S$, and $E_R \cup E_S = E$. (Why?) Now consider the sets of indices $\mathbb{N}_R = \{n \,|\, a_n \in E_R\}$ and $\mathbb{N}_S = \{n \,|\, a_n \in E_S\}$ of elements of $E$ that are in $E_R$ and $E_S$, respectively. Then $\mathbb{N}_R \cup \mathbb{N}_S = \mathbb{N}$. Can both $\mathbb{N}_R$ and $\mathbb{N}_S$ have largest elements (say, $N_1$ and $N_2$)? No, since $N = \max(N_1, N_2)$ would have been the largest positive integer in that case. This is impossible since $N + 1 > N$ is also a positive integer. Thus, one of the the two sets (say, $\mathbb{N}_R$) is nonempty and has no largest element. Let $n_1$ be the smallest integer in $\mathbb{N}_R$, $n_2$ be the smallest integer in $\mathbb{N}_R$ larger than $n_1$, $n_3$ be the smallest integer in $\mathbb{N}_R$ larger than $n_2$, etc. In general, for each $i \geq 1$, let $n_{i+1}$ be the smallest integer in $\mathbb{N}_R$ larger than $n_i$. Define a map $f : E_R \to E_R - \{a_{n_1}\}$ by $f(a_{n_i}) = a_{n_{i+1}}$. Then it is easy to see that $f$ is a bijection, so $E_R \subseteq R$ is infinite, so $R$ is infinite. Contradiction. Thus, $R \cup S$ is finite.