Part A.

1. In each part, prove that the two statements are not logically equivalent. To do that, find counterexamples P and Q which make one of the statements true and the other false. Does one of the statements in each pair imply the other? If so, which implies which? *Hint:* Easy counterexamples will do.

   (a) $(\forall x)(P(x) \vee Q(x))$ and $(\forall x)(P(x)) \vee (\forall x)(Q(x))$, where $P(x)$ and $Q(x)$ are some logical statements whose truth depends on the value of variable x.

   (b) $(\exists x)(P(x) \wedge Q(x))$ and $(\exists x)(P(x)) \wedge (\exists x)(Q(x))$, where $P(x)$ and $Q(x)$ are some logical statements whose truth depends on the value of variable x.

   (c) $(\forall x)(\exists y)(P(x,y))$ and $(\exists y)(\forall x)(P(x,y))$, where $P(x,y)$ is some logical statement whose truth depends on the value of variables x and y. *Hint:* The first statement reads: "for every x there is a y such that $P(x,y)$ is true," while the second statement reads: "there is a y such that for every x, $P(x,y)$ is true."

2. Let S be a set. We call S *infinite* if it has a nonempty proper subset S_0 (i.e. $S_0 \subseteq S$, $S_0 \neq S$, $S_0 \neq \emptyset$) and there is a bijection (a 1-1 and onto function) $\varphi : S_0 \to S$. (So, $\mathbb{Z}$ and $\mathbb{N}$ are infinite sets.) We call S *finite* if S is not infinite (in other words, there is no bijection between S and any of its nonempty proper subsets).

   (a) Prove that the empty set, $\emptyset$, is finite. *Hint:* Proof by contradiction, takes ≤ 30 seconds.

   (b) Let S be a set and $T \subseteq S$. Prove that if S is finite then T is finite. *Hint:* Prove the contrapositive: if T is infinite then S is infinite.

   Remark: You might have a different intuition as to what "finite" and "infinite" means. However, here you can only use the definitions given in this problem.

3. A binary relation $\sim$ on a set S (i.e. a relation between two elements of S) is called an *equivalence relation* on S if it satisfies the following properties for all $a, b, c \in S$:

   Reflexivity: $a \sim a$.

   Symmetry: $a \sim b \implies b \sim a$.

   Transitivity: $(a \sim b \ \wedge \ b \sim c) \implies a \sim c$.

   Let S be a set, $a, b \in S$. Which of the following are equivalence relations, which are not, and why?

   (a) S is any set, $a \sim b$ iff $a = b$.

   (b) S is any set, $a \sim b$ iff $a \neq b$.

   (c) S is a set of integers, $a \sim b$ iff $a \leq b$.

   (d) S is a set of integers, $a \sim b$ iff $a - b$ is divisible by a fixed integer m.

(e) S is a set of lines in the plane, $a \sim b$ iff $a \parallel b$.

(f) S is a set of lines in the plane, $a \sim b$ iff $a \perp b$.

Part B.

1. Which of these are equivalence relations (see Problem A3), which are not, and why?

   (a) $S = \bigcup_{\alpha \in T} S_\alpha$ where all the S_α are mutually disjoint, nonempty sets (and index α ranges over some index set T), $a \sim b$ iff both a and b are in the same S_α.

   (b) S is the collection of all sets, $a \sim b$ iff there is a bijection between set a and set b.

2. (a) Consider the equivalence relation defined in Problem A3. What is wrong with the following proof that symmetry and transitivity imply reflexivity? "Let $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity (using $c = a$)."

   (b) Suggest an alternative property close to reflexivity such that symmetry and transitivity do imply this alternative property.

3. Let S be the smallest set such that for some element $e \in S$ there exists a bijection $\varphi : S \to S \setminus \{e\}$. Prove that $S \cong \mathbb{N}$ (i.e. there is a bijection between S and $\mathbb{N}$). *Hint:* Consider elements $e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots$, and prove they are all distinct.

   Remark: This proves that $\mathbb{N}$ is the smallest infinite set (up to a bijection).

4. For sets, use the definitions of finite and infinite of Problem A2 above. Let R and S be finite sets. Prove that $R \cup S$ is a finite set.

   Remarks:

   (a) Results and proofs of Problems A2 and B3 may be useful here.

   (b) Try a proof by contradiction. Say both R and S are finite (as defined in problem A2), but $T = R \cup S$ is infinite (again, as defined in problem A2). Then there is a nonempty proper subset $T_0 \subseteq T$ such that there is a bijection $\varphi : T \to T_0$. Now $T_0 \subsetneq T$, so there is an element $e \in T \setminus T_0$. Consider elements $e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots$ in T. Each of them is either in R or in S. Now let $E = \{e, \varphi(e), \varphi(\varphi(e)), \varphi(\varphi(\varphi(e))), \ldots\}$, $E_R = E \cap R$ and $E_S = E \cap S$. Prove that at least one of E_R or E_S must be infinite, then prove that at least one of R or S must be infinite. Now deduce the result of the problem.

   (c) You probably have an intuition that a finite set is one with, say, n elements. In that case, if R has n elements and S has m elements, then $R \cup S$ ought to have $\leq n + m$ elements. If you actually wish to use that intuition here, you need to establish by strict argument that *your* notion of finite (i.e. having n elements) is logically the same as the notion of Problem A2. This is not so easy. Even if you do this, you are faced with the problem of $R \cup S$ and connection with addition in $\mathbb{N}$. At some point or points in your argument, some real ingenuity would be required.