Math 301 **Final Exam** **Due 05/06/04** **Name**______________________

This exam is **due Thursday, May 6, by 5pm, in my office, Carver 456**. You may consult the text for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. Answers to all questions except multiple choice require a rigorous proof. Show your argument *neatly*. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

Part A.

This part consists of multiple-choice questions. Please *circle* the answer which is *always true* or *always correct*. Circle the *entire* answer, not just the letter labelling it. No work is required and no partial credit will be given for any part of this problem.

1. If G is a group of order 8740 and if H is a subgroup of G, then the following numbers are all possible as the number of cosets of H in G.

   (a) $\{76, 92, 95, 391\}$
   (b) $\{40, 76, 115, 475\}$
   (c) $\{5, 95, 101, 437\}$
   (d) $\{4, 76, 95, 1748\}$
   (e) $\{4, 95, 115, 2116\}$

2. Consider the group S_{35} of permutations of the set $\{1, 2, \ldots, 35\}$; so S_{35} is the symmetric group on 35 letters. Write H for the subgroup of all permutations $\sigma \in S_{35}$ such that $\sigma(9) = 9$ and $\sigma(17) = 17$. Then

   (a) $|S_{35} : H| = 1190$
   (b) $|S_{35} : H| = 2$
   (c) $|S_{35} : H| = 33$
   (d) $|S_{35} : H| = 33!$
   (e) $|S_{35} : H| = 1155$

3. Consider the group G consisting of all 2×2 upper triangular real matrices $\sigma = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ for which $\det(\sigma) = 1$ and $a > 0$, $c > 0$. Let G act on the set $\mathbb{R}^2$ according to the rule:

$$\sigma \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cy \end{bmatrix}$$

(In other words, the permutation of $\mathbb{R}^2$ associated with a matrix $\sigma \in G$ maps each point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the point $\sigma\mathbf{v}$.) Let $\mathbf{v} \in \mathbb{R}^2$ be such that the stabilizer of $\mathbf{v}$ is the subgroup

$$\text{stab}(\mathbf{v}) = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid \text{all } b \in \mathbb{R} \right\}.$$

Then $\mathbf{v}$ can be:

(a) any point on the x-axis.
(b) any point on the x-axis except $(0, 0)$.
(c) any point on the x-axis except $(0, 0)$, $(1, 0)$, $(-1, 0)$.
(d) any point in the right half-plane, i.e. where $x > 0$.
(e) any point in the left half-plane, i.e. where $x < 0$.

4. Given the same G and its action on $\mathbb{R}^2$ as in the previous problem:

(a) There are no points $(x, y) \in \mathbb{R}^2$ whose stabilizer is the identity subgroup of G.
(b) There are points, but only finitely many, whose stabilizer is the identity subgroup of G.
(c) Except for the points on a certain straight line, all other points have stabilizer the identity subgroup of G.
(d) The points whose stabilizer is the identity subgroup of G form a straight line.
(e) None of the above is a true statement.

5. The group $U(15)$ is isomorphic to:

(a) D_4.
(b) $\mathbb{Z}_8$.
(c) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.
(d) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
(e) None of the above.

Part B.

In this part, partial credit will be given. Show all work here. All answers require rigorous proofs.

1. Let $\phi : G \to G'$ be a homomorphism with $\ker \phi = K$, and let H be a subgroup of K. Find $\phi^{-1}(\phi(H))$.

2. Prove that $\mathbb{R}^\times = (\mathbb{R} - \{0\}, \times)$ (the multiplicative group of nonzero real numbers) and $\mathbb{R}^+ = (\mathbb{R}, +)$ (the additive group of real numbers) are not isomorphic.

   Hint: Consider the kernel of any homomorphism from $\mathbb{R}^\times$ to $\mathbb{R}^+$.

3. Let G be a nonabelian group of order 21. It can be proved (and you may assume this) that G must have a unique subgroup H of order 7, which is normal in G.

   (a) Let $a, b \in G$ be such that $|a| = 3$ and $|b| = 7$ (they exist by Cauchy theorem, assume this). Prove that $G = \langle a, b \rangle$. Determine the possible values of aba^{-1} (so that it is of the form $a^i b^j$ for some $0 \le i \le 2$ and some $0 \le j \le 6$).

   Hint: There are two answers. Determine which part of G aba^{-1} should be in, then consider $aaaba^{-1}a^{-1}a^{-1}$.

   (b) Prove that all nonabelian groups of order 21 are isomorphic to each other. (In other words, there is only one such group up to an isomorphism.)

   Hint: Think about how the two answers in part (a) are related.

4. Define the *cycle structure* of a permutation as the list of the lengths of its cycles (except 1-cycles) in the cycle notation (with multiplicities) in non-increasing order. [E.g. (12)(385)(49) has cycle structure $(3, 2, 2)$.] Prove that any two permutations in S_n have the same cycle structure if and only if they are conjugate.

   Hint: This is not unlike what a change of basis does to matrices of linear transformations, only here instead of basis vectors, we have elements $1, 2, \ldots, n$.

5. Let G be a finite group, and write $c(G)$ for the number of conjugacy classes of G. In general, $c(G)$ will increase as $|G|$ increases, so we introduce the number $\gamma(G) = c(G)/|G|$.

   (a) Prove that $0 < \gamma(G) \le 1$, and $\gamma(G) = 1$ if and only if G is abelian.

   (b) Now assume G is nonabelian. Prove that $\gamma(G) \le 5/8$.

   Hint: Take the Class Equation $|G| = |Z(G)| + \sum |G : C(a)|$, where the sum is over representatives of conjugacy classes not in $Z(G)$ (see Ch. 24, pp. 397-399), and divide through by $|G|$. Consider the center $Z(G)$ of G and the rest of G.

If G is nonabelian, what is the maximum possible size of $Z(G)$? (Theorem 9.3 may help.) What is the minimum possible size of each conjugacy class not in $Z(G)$? (I.e. what is the smallest possible size of a summand in the $\sum |G : C(a)|$ part of the above sum?) What is the maximum possible number of the conjugacy classes not in $Z(G)$? (I.e., what is the largest possible number of summands in the $\sum |G : C(a)|$ part of the above sum?) Don't forget to prove your answers to each question.

(c) More generally, if G is nonabelian and p is the smallest prime which divides $|G|$, prove that

$$\gamma(G) \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

Hint: Same as for part (b).

(d) (extra) Are the above bounds sharp? That is, can you find a group G with $\gamma(G) = 5/8$? similarly, for (c)?