Planarity

Math 314

April 17, 2003

1 Planar graphs

Definition 1.1 A *planar* graph is a graph that can be drawn in the plane without edge crossings. Such a drawing of a planar graph is called a *plane drawing*.

Note that adjacent edges are not considered to cross at their common vertex. Note that a graph without any edges is planar because, obviously, no two of its edges cross in any drawing. Finally, note that a graph is nonplanar if *every* drawing of it in the plane has to contain at least one edge crossing.

Example 1.2 Some examples of planar graphs:

1. All trees (including all paths).
2. All cycles.
3. All wheels.
4. All double wheels.
5. Some complete graphs: K_0, K_1, K_2, K_3, K_4.
6. Some complete bipartite graphs: $K_{0,n} \cong K_{n,0}, K_{1,n} \cong K_{n,1}, K_{2,n} \cong K_{n,2}$.

Example 1.3 Some examples of nonplanar graphs: K_5, $K_{3,3}$, Petersen graph, $C_3 \times C_3$.

Planarity is often easier to prove than nonplanarity. To show that a graph is planar, it is enough to exhibit a drawing of the graph in the plane without edge crossings. On the other hand, to show that a graph is nonplanar, we must prove that *all* possible drawings of the graph in the plane will contain an edge crossing somewhere. Therefore, we usually have to find some property of the graph which would make a drawing without edge crossings impossible.

Definition 1.4 A *region*, or a *face*, in a plane drawing is a subset of the plane bounded by a cycle without diagonal paths.

The term "face" comes from geometry: any 3-dimensional polyhedron can be projected onto a plane as a plane drawing of some planar graph. The regions of that plane drawing correspond to the faces of the polyhedron.

One of the regions is always unbounded (has infinite area). It is called the *exterior* region, or the *outer* region.

It turns out that any plane drawing of the same planar graph must have the same number of regions. In fact, a fundamental result of Leonhard Euler connects the numbers of vertices, edges and regions for a connected planar graph. That result was originally applied to 3-dimensional polyhedra, and is therefore called Euler's Polyhedral Formula.

Theorem 1.5 (Euler's Polyhedral Formula) *For any plane drawing of a connected planar graph G with p vertices, q edges and r regions, we have $p - q + r = 2$.*

PROOF. We will prove this theorem by induction on the number of regions of G.

Induction Base. Suppose $r = 1$. Then G has no cycle (if G had a cycle, it would have at least 2 regions: 1 inside the cycle and 1 outside the cycle). Since G is connected and acyclic, G must be a tree, so $q = p - 1$. Therefore, $p - q + r = p - (p - 1) + 1 = 2$ as desired.

Induction Step. Let $r > 1$. Assume the theorem is true for all plane drawings of all planar graphs with $< r$ regions. Consider any plane drawing of a planar graph G that has r regions. Since $r > 1$, G must have a cycle. Let e be any edge of that cycle. Then we cannot have the same region on both sides of e, so e must separate two distinct regions. Therefore, by removing the edge e, we will merge two regions of G into one. Consider the graph $G - e$. It has p vertices, $q - 1$ edges and $r - 1$ regions. Since $r - 1 < r$, our inductive hypothesis applies to $G - e$, so $p - (q - 1) + (r - 1) = 2$, i.e. $p - q + r = 2$. Thus, the theorem is true for any plane drawing of any planar graph with r regions.

Thus, by induction, the theorem is true for all plane drawings of all planar graphs G. □

It is intuitively clear that if we start with some vertices and no edges (which is a planar graph), and keep adding edges between existing vertices while trying to keep the graph planar, until we get to a complete graph, then either the resulting complete graph is planar, or somewhere along the way we will arrive at a planar graph such that any additional edge between its vertices would make it nonplanar. We will now consider these graphs, called maximal planar graphs.

Definition 1.6 A *maximal planar* graph is a planar graph such that addition of any edge makes it nonplanar (or is impossible, if the graph is also complete).

Obviously, a maximal planar graph must be connected, but there are other conditions it must also satisfy.

Example 1.7 Some examples of maximal planar graphs: K_0, K_1, K_2 (these three are special because they are very small and don't have too many edges to begin with), $K_3, K_4, W_3 \cong K_4, D_3 \cong K_5 - edge$.

The following theorem gives a characterization of maximal planar graphs.

Theorem 1.8 *A connected graph with ≥ 3 vertices is maximal planar if and only if every region of every plane drawing of it is a triangle.*

PROOF. ($\Leftarrow$) If some region is not a triangle, then we can cut that region into 2 with an additional edge between a pair of nonadjacent vertices on the cycle which bounds that region. Therefore, if some region in some plane drawing is not a triangle, then the graph is not maximal planar. Thus, if the graph is maximal planar, then every region of every plane drawing of it is a triangle.

($\Rightarrow$) If a graph G is planar, but not maximal planar, then we can add a new edge e between two vertices of G so that $G \cup e$ is still planar. Consider a plane drawing of $G \cup e$. Remove the edge e from it. Then the two regions separated by e will merge into one, and the endpoints of e are nonadjacent. Therefore, the new merged region must be bounded by a cycle of ≥ 4 vertices (otherwise all vertices on that cycle will be mutually adjacent), hence that region is not a triangle. Therefore, if the graph is not maximal planar, then some region of some plane drawing of it is not a triangle. Thus, if every region of every plane drawing of it is a triangle, then the graph is maximal planar. □

Note that the theorem above implies that a maximal planar graph on $\geq 4 vertices$ cannot have vertices of degree < 3.

This criterion lets us find a nice and simple necessary (but not sufficient!) condition for a graph to be maximal planar.

Theorem 1.9 *A maximal planar graph with $p \geq 3$ vertices has $q = 3p - 6$ edges.*

Proof. Add up all edges in each region, then sum up over all regions. Each of r regions is bounded by 3 edges, so the total sum is $3r$. On the other hand, each of the q edges in that sum is counted twice: once for each of the two regions it separates. Therefore, the total sum is equal to $2q$. Thus, $3r = 2q$, so $r = 2q/3$. Substituting this into the Euler's formula $p - q + r = 2$, we can easily obtain $q = 3p - 6$. Note that it also implies $r = 2p - 4$. □

Remark 1.10 *Caution:* Note that the converse of this theorem is false: a graph with $p \geq 3$ vertices and $q = 3p - 6$ edges does not have to be maximal planar or even just planar.

Theorem 1.9 has a nice corollary that applies to all planar graphs.

Theorem 1.11 *A planar graph with $p \geq 3$ vertices has $q \leq 3p - 6$ edges.*

Proof. Consider any planar graph G with p vertices. Start adding edges between vertices of G which will keep the resulting graph planar. After a certain step in this process it will be impossible either to add any more edges or to add any edges that will keep the resulting graph planar. This means that the graph we finally obtained is maximal planar, so it has $3p - 6$ edges. Since this graph has at least the q edges of G, it follows that $q \leq 3p - 6$. □

Corollary 1.12 *If a graph on $p \geq 3$ vertices has $q > 3p - 6$ edges, then it is nonplanar.*

Remark 1.13 *Caution:* Note that the converse of Theorem 1.11 and 1.17 is false: a graph with $p \geq 3$ vertices and $q \leq 3p - 6$ edges may be nonplanar. For example, $K_{3,3}$ is a nonplanar graph (as we will see later) with $p = 6$ vertices and $q = 9$ edges, and $9 \leq 3 \cdot 6 - 6$.

Example 1.14 K_5 is nonplanar, since it has $p = 5$ vertices and $q = 10$ edges, and $10 > 9 = 3 \cdot 5 - 6$.

Corollary 1.15 *If a graph is planar, then it has at least one vertex of degree at most 5. In other words, if G is planar, then the minimal degree $\delta(G) \leq 5$.*

Proof. If $\delta(G) \geq 6$, i.e. if every vertex of G has degree at least 6, then the sum of the degrees $2q \geq 6p$, i.e. $q \geq 3p > 3p - 6$, so G is nonplanar. But that contradicts the fact that G is planar, so $\delta(G) \leq 5$. □

We can prove that $K_{3,3}$ is nonplanar, by taking into account that it is also bipartite and considering bipartite planar graphs. The following theorem, similar to Theorem 1.11, establishes a upper bound on the number of edges q of a bipartite planar graph.

Theorem 1.16 *A bipartite planar graph on $p \geq 3$ vertices and q edges has $q \leq 2p - 4$.*

Proof. Let G be a bipartite planar graph. Consider a plane drawing of G. Any region in that plane drawing has ≥ 4 sides since C_4 is the shortest even cycle, hence the shortest cycle a bipartite graph can have. Therefore, the sum, over all r regions, of the numbers of sides bounding each region is $\geq 4r$. But each edge is counted twice in that sum, so it is equal to $2q$. Hence, $2q \geq 4r$, i.e. $r \leq q/2$. Therefore, $2 = p - q + r \leq p - q + q/2 = p - q/2$, i.e. $q \leq 2p - 4$. □

Corollary 1.17 *If a bipartite graph on $p \geq 3$ vertices has $q > 2p - 4$ edges, then it is nonplanar.*

Example 1.18 $K_{3,3}$ is nonplanar, since it has $p = 6$ vertices and $q = 9$ edges, and $9 > 8 = 2 \cdot 6 - 4$.

Thus, K_5 and $K_{3,3}$ are both nonplanar. In fact, any graph that contains K_5 or $K_{3,3}$ as a subgraph is also nonplanar. This statement can be generalized as follows.

Definition 1.19 A *subdivision* of a graph results from adding some (possibly none) vertices of degree 2, each "splitting" an edge. In other words, each edge is replaced by a path between the endpoints of that edge so that the entire collection of these paths is internally disjoint (no interior vertex of any path occurs on any other path).

Clearly, a graph is planar if and only if any subdivision of it is planar. Also, it is easy to see from the above that any graph that contains a subdivision of K_5 or $K_{3,3}$ as a subgraph is also nonplanar. In other words, if a graph is planar, then it has no subgraph which is a subdivision of K_5 or $K_{3,3}$. But what about the converse? Does a graph with no subdivision of K_5 or $K_{3,3}$ as a subgraph have to be planar? A Polish mathematician Kuratowski proved this in the 1930s.

Theorem 1.20 (Kuratowski) *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$ as a subgraph. Equivalently, a graph G is nonplanar if and only if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

The proof of Kuratowski's Theorem is beyond the scope of this course, and we will omit it.

Using this theorem we can show, for example, that Petersen graph and $C_3 \times C_3$ are nonplanar graphs.

Finally, we note another couple of properties of planar graphs.

Theorem 1.21 (Wagner-Fary) *If a graph is planar, then in can be drawn so that all edges are straight lines.*

Theorem 1.22 *If a graph G is maximal planar graph on $p \geq 4$ vertices, and p_i is the number of vertices of degree i ($i \geq 0$), then $3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + 3p_9 + 4p_{10} + \dots$.*

Proof. Note that $p_1 = p_2 = 0$, $p = \sum_i p_i$, $2q = \sum_i ip_i$, and $q = 3p - 6$. The conclusion follows. □

2 Planar maps

Definition 2.1 A *map* is a plane drawing of a connected bridgeless planar multigraph. A region of a map is called a *country*.

Remark 2.2 Note that the same (multi)graph may have many different drawings. The multigraph above needs to be connected so that $p - q + r = 2$ and bridgeless so that each edge separates two countries (i.e. so that no country can be on both sides on any edge). Since it is a multigraph, we allow multiple edges and 2-cycles (called *lunes*).

We will mostly consider maps of special type, where no 4 countries can have a common vertex.

Definition 2.3 A *normal map* is a map whose underlying graph is cubic (i.e. 3-regular).

In other words, a normal map is a map where every vertex has degree 3.

A longstanding problem (1850s-1970s) in this area was the so called Four Color Problem: *can the countries of a normal map be colored with at most 4 colors so that adjacent countries have different colors?* This problem was finally solved by K. Appel and W. Haken by reducing the proof to 1500-2000 cases and analyzing them with the help of a computer. Since then the number of cases has been greatly reduced, but it's still quite a lot, and the proof is still pretty long.

Theorem 2.4 (Four Color Theorem) *The countries of a normal map can be colored with* ≤ 4 *colors so that adjacent countries have different colors.*

Note that the exterior region is also a country and must also be colored.

An earlier attempt by Kempe to prove the Four Color Theorem seemed to be successful until Heawood discovered a flaw in its proof. Heawood could still extract the proof of the weaker Five Color Theorem from it (replace 4 with 5 in Theorem 2.4).

This theorem can be restated in more familiar terms with the help of the following operation.

Definition 2.5 The *dual* of a planar graph is formed by placing a vertex in each region and joining all pairs of vertices in adjacent regions through every edge that separates these regions.

Notation 2.6 The dual of a graph G is denoted G'.

Theorem 2.7 *Let* G *be a planar map. Then the following statements are true:*

1. G *is a planar map* $\iff$ G' *is a planar map.*
2. $(G')' = G$.
3. *There is a 1-1 correspondence between:*
   (a) *vertices of* G' *and regions of* G*;*
   (b) *edges of* G' *and edges of* G*;*
   (c) *regions of* G' *and vertices of* G*.*
4. G *has* p *vertices,* q *edges and* r *regions* $\iff$ G' *has* r *vertices,* q *edges and* p *regions.*

Using the operation of taking the dual, we can reformulate some of our earlier results. The proofs are only sketched.

Theorem 2.8 *No planar map has 5 mutually adjacent countries.*

Proof. The dual graph is planar, hence cannot contain a K_5. □

Theorem 2.9 *Every planar map has a country with* ≤ 5 *sides.*

Proof. The dual graph is planar, hence has a vertex of degree ≤ 5. □

Theorem 2.10 *The dual of a normal map is a maximal planar graph, and vice versa (with ≥ 3 vertices).*

Proof. A normal map has all vertices of degree 3. A maximal planar graph has all regions with 3 sides. □

Using Theorems 2.7 and 2.10, we can reformulate the Four Color Theorem as follows by taking the dual.

Theorem 2.11 *For every maximal planar graph G, $\chi(G) \leq 4$.*

In fact, since every planar graph is a subgraph of some maximal planar graph, it follows that

Theorem 2.12 (Four Color Theorem) *For every planar graph G, $\chi(G) \leq 4$.*

In 1880, Tait found the following corollary of the Four Color Theorem.

Theorem 2.13 (Tait) *For every cubic (3-regular) bridgeless planar graph G, $\chi'(G) \leq 3$.*

The outline of the proof is as follows.

Proof. G is 3-regular, so $\chi'(G) \geq 3$. Color the countries of G with 4 colors: red (R), blue (B), green (G) and yellow (Y). Color edges between the countries as follows:

Colors of countries	Colors of edges
RB or GY	1
RG or BY	2
RY or BG	3

Since $\Delta(G) = 3$, it follows that 3 countries can have at most a vertex in common, while 4 countries can have no point in common, so our edge coloring is proper. □