1. Complete the table by finding all nonnegative integers m and n for which the graphs below have the following properties:

	bipartite	n-connected	$\chi = n$	$\chi' = n$
C_n	$n \geq 3$	$n = 3$	$n = 3$	$n = 3$
$K_{m,n}$	$m, n \geq 0$	$m \geq n \geq 0$; or $(m,n) = (0,1)$	$n = 2$, $m \geq 1$; or $(m,n) = (0,0), (0,1)$	$n \geq m \geq 1$; or $(m,n) = (0,0)$
K_n	$n \leq 2$	$n \geq 0$	$n \geq 0$	odd $n \geq 3$; or $n = 0$

2. In a connected graph with a 2-coloring of the vertices, two vertices x and y have the same color if and only if **the distance $d(x,y)$ between x and y is even**.

3. (a) K_4 is a cubic (3-regular) graph with $\chi(K_4) = 4 > 3\chi'(K_4)$.

   (b) $\overline{K_2}$ (two isolated vertices) is a disconnected graph where $\chi(\overline{K_2}) = 1 > 0 = \Delta(\overline{K_2})$.

4. Let $S_j = K_{1,j}$ denote the star with j leaves. Let us prove that the Ramsey number

$$r(S_m, S_n) = \begin{cases} m+n-1 & \text{if both } m \text{ and } n \text{ are even,} \\ m+n & \text{otherwise.} \end{cases}$$

In a coloring *without* either a red S_m and a blue S_n, every vertex has $\leq m-1$ red edges and $\leq n-1$ blue edges incident with it. So each vertex has $\leq m+n-2$ neighbors and the number of vertices is $\leq m+n-1$. Thus, $r(S_m, S_n) \leq m+n-1+1 = m+n$.

Case 1. If $m+n-1$ is even, then there is an $(m-1)$-regular graph on $m+n-1$ vertices. Take a 1-factor of K_{m+n-1} and use a turning trick $m-1$ times to get an $(m-1)$-regular red subgraph. Then continue with the turning trick and use it $n-1$ more times to get an $(n-1)$-regular blue subgraph. Hence, in this case $r(S_m, S_n) > m+n-1$, so $r(S_m, S_n) = m+n$.

Case 2. If $m+n-1$ is odd, but both $m-1$ and $n-1$ are even, then there is an $(m-1)$-regular red subgraph and an $(n-1)$-regular blue subgraph of K_{m+n-1} since K_{m+n-1} is decomposable into Hamilton cycles (which are 2-factors of K_{m+n-1}). Start with a Hamilton cycle C_{m+n-1} of K_{m+n-1} and use a turning trick $(m-1)/2$ times for the red subgraph, then the remaining $(n-1)/2$ times for the blue subgraph. Hence, in this case $r(S_m, S_n) > m+n-1$, so $r(S_m, S_n) = m+n$ as well.

Case 3. If $m+n-1$ is odd and both $m-1$ and $n-1$ are odd, then there is no $(m-1)$-regular or $(n-1)$-regular graph on $m+n-1$ vertices since the sum of the degrees in such a graph would have been odd. Hence, in this case we cannot avoid both a red K_m and a blue K_m. Therefore, $r(S_m, S_n) \leq m+n-1$ in this case. However, $m+n-2$ is even, so Case 1 applies, and hence, $r(S_m, S_n) > m+n-2$, so $r(S_m, S_n) = m+n-1$ when both m and n are even.

5. $K_{4,4}$ may be decomposed into 2 Hamilton cycles C_8. If one is colored red and the other blue, we will have no monochromatic $K_{2,2}$ (i.e. C_4).

6.

$$\chi_{\text{new}}(C_n) = \begin{cases} 3, & \text{if is } n \text{ is a multiple of 3,} \\ 5, & \text{if } n = 5, \\ 4, & \text{otherwise.} \end{cases}$$

To see this, note that the first 3 vertices must be colored in 3 different colors, hence, $\chi_{\text{new}}(C_n) \geq 3$. If $n = 3k$ for some integer $k \geq 1$, then it is easy to see that $123123\ldots123$ is a valid coloring (check!). Note that this is the *only* way (up to renaming colors) to color C_n in 3 colors so that vertices with distance at most 2 from each other receive different colors. Hence, it is not a valid coloring when $n \neq 3k$, so for those n, $\chi_{\text{new}}(C_n) \geq 4$. If $n = 3k+1$ for some integer $k \geq 1$ (i.e. $n = 4, 7, 10, \ldots$), then again it is easy to see that $4123123\ldots123$ is a valid coloring (check!), so $\chi_{\text{new}}(C_n) \leq 4$. If $n = 3k+2$ for $k \geq 2$ (i.e. $n = 8, 11, 14, \ldots$), then $41234123123\ldots123$ is a valid coloring (check!), so $\chi_{\text{new}}(C_n) \leq 4$. If $n = 5$, then the distance between any two vertices is either 1 or 2 (going clockwise or counterclockwise, whichever is shorter). Hence, $\chi_{\text{new}}(C_5) \geq 5$, but coloring 12345 is valid, so $\chi_{\text{new}}(C_5) \leq 5$.

7. Let us prove by induction that $\kappa(Q_n) = n$ for $n \geq 2$. Since, $Q_2 \cong C_4$, we have $\kappa(Q_2) = \kappa(C_4) = 2$, so the base case is true. For the induction step, let us now assume that the statement is true for Q_{n-1}, i.e. that $\kappa(Q_{n-1}) = n-1$. Since Q_n is n-regular, $\kappa(Q_n) \leq \delta(Q_n) = n$. Hence, we only need to prove that $\kappa(Q_n) \geq n$.

   Consider two subgraphs of Q_n: the bottom $Q_{n-1} \times \{0\}$ and the top $Q_{n-1} \times \{1\}$. Choose a set U of $n-1$ vertices in Q_n to be deleted. We distinguish two cases.

   Case 1. Suppose the $n-1$ vertices in U are all in the top subgraph. Then the bottom is left untouched in $Q_n - U$, hence the bottom remains connected. Moreover, all remaining top vertices are connected to their bottom counterparts. Therefore, $Q_n - U$ is connected. The case where $n-1$ vertices in U are all in bottom subgraph is very similar.

   Case 2. Suppose some vertices in U are in the top subgraph and some are in the bottom subgraph. That means $< n-1$ vertices were deleted from the top and from the bottom, so the remainders of the top and of the bottom are still connected by the induction hypothesis that $\kappa(Q_{n-1}) = n-1$. But are they connected to each other? Notice that Q_{n-1} has $2^{n-1} > n-1$ vertices, so no matter which $n-1$ vertices we delete from Q_n, there must be a vertex x in Q_{n-1} so that neither $(x, 0)$ nor $(x, 1)$ are in U. Therefore, the remains of the top and the bottom in $Q_n - U$ are connected at least by the edge between $(x, 0)$ and $(x, 1)$. Thus, in $Q_n - U$, the top is connected, the bottom is connected, and the top is connected to the bottom, so the whole $Q_n - U$ is connected.

   In both cases, $Q_n - U$ is connected for any set U of $n-1$ vertices, so $\kappa(Q_n) \geq n$. QED.

8. Let us prove that a regular graph which is decomposable into spanning trees must be complete. Let G be an r-regular graph with p vertices and q edges, which is decomposable into k spanning trees. Each of p vertices of G has degree r, so the sum of all degrees is $rp = 2q$. On the other hand, each spanning tree of G also has p vertices, hence $p-1$ edges. If G is decomposable into k spanning trees, then $q = k(p-1)$. Therefore, $rp = 2k(p-1)$, so $r = rp - r(p-1) = 2k(p-1) - r(p-1) = (2k-r)(p-1)$, so r is a multiple of $p-1$. But $0 \leq r \leq p-1$, so $r = 0$ or $r = p-1$. Therefore, $p = 2k(p-1)/r = 2k$, i.e. p is even and $k = p/2$. If $r = 0$, then G has no edges and is disconnected, hence cannot be decomposable into spanning trees. Therefore, $r = p-1$, i.e. G is complete. We also get that $p = 2k(p-1)/r = 2k$, i.e. p is even and $k = p/2$.

9. (a) Let C be the longest cycle in a graph and suppose there exists a vertex w not on the cycle. If w had two consecutive neighbors on the cycle, say x and y, then replacing the edge xy with xw and wy would result in a longer cycle than C. Therefore, w cannot have two consecutive neighbors on the cycle.

   (b) Pick an orientation on C (i.e. a direction in which to travel along C). Let S denote the set of successors of neighbors of w on C. Suppose that two vertices on S are adjacent, say x' and y', successors of neighbors x and y of w. Without loss of generality, assume that in the direction we travel along C, we encounter x (then x'), then y (then y'). Then the cycle $C' = xwy(\text{back along } C \text{ to})x'y'(\text{forward along } C \text{ to})x$ obtained by deleting xx' and yy' and inserting xw, wy and $x'y'$, is a longer cycle than C, which contradicts our assumption. Therefore S is an independent set.