Math 314 Midterm 2 Due 04/08/2003

This exam is due Tuesday, April 8, in class. You may consult the text, handouts and homework solutions for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. Complete the table by finding all nonnegative integers m and n for which the graphs below have the following properties:

	bipartite	n-connected	$\chi = n$	$\chi' = n$
C_n				
$K_{m,n}$				
K_n				

2. Complete the following: "In a connected graph with a 2-coloring of the vertices, two vertices x and y have the same color if and only if ________________________."

3. Give examples:

   (a) A cubic graph G with $\chi(G) > \chi'(G)$.

   (b) A disconnected graph where the chromatic number is more than the maximum degree.

   Hint: Simple examples will do.

4. *(extra credit)* Let $S_j = K_{1,j}$ denote the star with j leaves. Prove that the Ramsey number

$$r(S_m, S_n) = \begin{cases} m+n-1 & \text{if both } m \text{ and } n \text{ are even,} \\ m+n & \text{otherwise.} \end{cases}$$

   Hint: $p = r(S_m, S_n)$is the smallest integer such that every red-and-blue coloring of the complete graph K_p with must either have a vertex with red degree $\geq m$ or a vertex with blue degree $\geq n$. Try to find the upper bound on p. Then show that there is a 2-coloring of K_{p-1} that does not have this property.

5. Color the edges of $K_{4,4}$ by two colors so that there is no monochromatic $K_{2,2}$.

6. Color the vertices of C_n in such a way that any two vertices with distance ≤ 2 have different colors. How many colors are required? *Hint:* The answer depends on whether n is divisible by 3. There is also one small special case.

7. Prove that $\kappa(Q_n) = n$. *Hint:* There are several ways of doing this. You can do it directly from definition, or use Whitney's theorem, or use induction, or combine some of the above techniques.

8. Prove that a regular graph which is decomposable into spanning trees must be complete. Express the number of spanning trees in such a decomposition using the number of vertices of the graph. *Hint:* It's all in the numbers. Let G be an r-regular graph with p vertices and q edges, which is decomposable into k spanning trees. Express q in two ways using these parameters. Use a divisibility argument to prove that $r = p - 1$.

9. *(extra credit)* A set of vertices is called *independent* if no two vertices in the set are adjacent. Prove that if a graph is n-connected ($n \geq 2$) and every independent set has size $\leq n$, then the graph is Hamiltonian.

   Hint: You might wish to proceed as follows:

   (a) Let C be the longest cycle in a graph and suppose there exists a vertex w not on the cycle. Prove that w cannot have two consecutive neighbors on the cycle.

   (b) Pick an orientation on C. Let S denote the set of successors of neighbors of w on C. Show that S is an independent set.

   (c) Prove that the above results generalize if we consider internally disjoint paths from w to C rather than just edges.