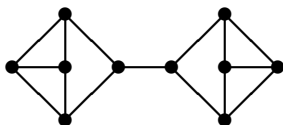


1. The answers are as follows:

	cubic	acyclic	Eulerian	Hamiltonian
C_n	never	never	$n \geq 3$	$n \geq 3$
$K_{m,n}$	$m = n = 3$	$m = 0, 1$ or $n = 0, 1$	even $m, n \geq 2$; or $m + n = 1$	$m = n \geq 2$; or $m + n = 1$
K_n	$n = 4$	$n = 0, 1, 2$	odd n	$n = 1$ or $n \geq 3$

2. A graph such that there is at most one path between every two vertices is a **forest**. [Note “at most one” (≤ 1) rather than “exactly one” ($= 1$) as in a tree.]
3. (a) The 4-cycle C_4 is the smallest example of a graph in which every 3 vertices are in a common cycle, but which can be disconnected by removing only 2 vertices.
 (b) Here is perhaps the simplest example of a cubic graph which is not Hamiltonian. (Petersen graph is another such example.)



This graph is not Hamiltonian because it has a bridge (the center edge, in this case), i.e. can be disconnected by removal of just one edge. However, at least 2 edges are required to disconnect a Hamiltonian cycle, let alone a graph that contains a Hamiltonian cycle.

4. The non-internal vertices in a tree must have degree < 2 , i.e. ≤ 1 . Since a tree has no isolated vertices, it means that non-internal vertices have degree 1, i.e. non-internal vertices are simply leaves. There are i internal vertices out of the total of p vertices, so there are $p - i$ leaves. A tree on p vertices has $p - 1$ edges, so the sum of the degrees of all vertices is $2(p - 1) = 2p - 2$. But the sum of all degrees is the sum of all internal degrees plus the sum of degrees of all leaves, i.e.

$$\sum_{v \in I} \deg v + (p - i) \cdot 1 = 2p - 2,$$

which is equivalent to

$$\sum_{v \in I} \deg v = p + i - 2.$$

5. We know that the degree sequence of our graph is $54321x$, where x is the unknown degree. Since the graph has 6 vertices, $x \leq 6 - 1 = 5$. Since we must have an even number of odd degrees, and have three odd degrees already, x must be odd. That means $x = 1$ or $x = 3$ or $x = 5$. Using the Havel-Hakimi procedure for each of the three cases, we find that only 543321 is a graphic sequence, so $x = 3$.
6. Each of the 6 vertices of K_6 has degree $6 - 1 = 5$. If K_6 has q edges, then $2q = 6 \cdot 5$, so $q = \frac{6 \cdot 5}{2} = 15$. Since 15 is odd, K_6 is not decomposable into P_2 . On the other hand, 6 is even, so K_6 is decomposable into Hamilton paths, and a Hamilton path of K_6 is P_5 , so K_6 is decomposable into P_5 .
7. As we know, an edge is a bridge if and only if it lies on no cycle. A tree has no cycle, so no edge lies on any cycle, so every edge is a bridge. Also, a tree is connected. Conversely, if every edge of a graph is a bridge, then no edge of the graph lies on any cycle. Therefore, the graph has no cycle. Since it is also connected, it must be a tree.

8. Decompose K_8 into two graphs on 8 vertices (call them 0, 1, 2, 3, 4, 5, 6, 7 and place them clockwise on a picture) as follows: G consists of the cycle 036147250 (skip 2 vertices clockwise) and the edges 04, 15, 26, 37 (skip 3 vertices); and \overline{G} consists of the cycles 012345670 (skip 0 vertices), 02460 and 13571 (skip 1 vertex). Then G does not contain K_3 as a subgraph and \overline{G} does not contain K_4 as a subgraph. Therefore, $r(3, 4) > 8$.
9. (a) The degree sequence of W_n is $n, 3, 3, \dots, 3$ (where 3 repeats n times). The hub vertex has degree n and the vertices on the cycle have degree 3. W_n has $n + 1$ vertices and $n + n = 2n$ edges (n cycle edges and n spokes).
- (b) W_n is connected since all vertices are connected to the hub vertex. W_n is never Eulerian since the vertices on the cycle have odd degree 3. W_n is always Hamiltonian (for $n \geq 3$), one example of a Hamilton cycle is $12 \dots nv1$, where v is the hub vertex, and $1, 2, \dots, n$ are the vertices on the cycle. W_n is always decomposable into P_2 since it has an even number of edges.
- (c) $W_3 \cong K_4$ is decomposable into 2 P_3 's. W_4 is decomposable into 2 P_4 's.
- (d) Given any decomposition of W_n (where $n \geq 7$ is odd) into two subgraphs, the larger of the degrees of the hub vertex in the two subgraphs must be $> n/2$ (it cannot be equal to $n/2$ since n is odd). Then in the other subgraph the degree of the hub vertex is $< n/2$ and the degrees of the cycle vertices are $\leq 3 < n/2$. Thus, the degree sequences of the two graph in the decomposition must be different, so the graphs cannot be isomorphic.
- (e) The case of W_5 is similar, but with an additional wrinkle. The larger of the degrees of the hub in the two subgraphs must be $> 5/2$, but it may be equal to 3, so if the two graphs are isomorphic then the hub vertex in one subgraph must correspond to a cycle vertex in the other. The degree sequence of W_5 is 533333. Suppose it splits into $x_0x_1x_2x_3x_4x_5$ and $y_0y_1y_2y_3y_4y_5$, where x_0 and y_0 are the degrees of the hub. The graphs are isomorphic, so their degree sequences must be rearrangements of each other. Then $x_0 + y_0 = 5$, and $x_i + y_i = 3$ for $i = 1, 2, 3, 4, 5$. Without loss of generality, assume $x_0 = 3$, then $y_0 = 2$. One of y_i 's must be 3, say y_1 so $x_1 = 0$; and one of x_i 's must be 2, say $x_2 = 2$, so $y_2 = 1$. Thus, we have degree sequences $302x_3x_4x_5$ and $231y_3y_4y_5$. So, one of the remaining y_i 's is 0, say $y_3 = 0$, so $x_3 = 3$; and one of the x_i 's is 1, say $x_4 = 1$, then $y_4 = 2$. Thus, we have $30231x_5$ and $23102y_5$. Now first sequence has an extra 3, and the second sequence has an extra 2, so $x_5 = 2$, $y_5 = 3$. But then $x_5 + y_5 = 5 \neq 3$. Contradiction. Therefore, W_5 is not decomposable into two isomorphic subgraphs. [Taken from Jason Hermann's exam.]
10. (a) The degree sequence of D_n is $n, n, 4, 4, \dots, 4$ (where 4 is repeated n times). D_n has $n + 2$ vertices and $n + n + n = 3n$ edges (n spokes from each of the two hubs, and n edges on the cycle C_n).
- (b) D_n is connected since each hub vertex is attached to every vertex on the cycle. D_n is Hamiltonian, to find a Hamilton cycle, replace one edge of the C_n subgraph by two edges from one hub, and another edge of the C_n by two edges from the other subgraph. D_n is Eulerian if and only if all degrees are even. But 4 is always even, so D_n is Eulerian if and only if n is even (and $n \geq 3$). D_n is decomposable into P_2 if and only if its number of edges, $3n$, is even, i.e. if and only if n is even.
- (c) Number the vertices on the cycle with $0, 1, 2, \dots, n - 1$. Now we can decompose D_n into n P_3 's (visualize them as zigzags): $u01v, u12v, \dots, u(n - 2)(n - 1)v, u(n - 1)0v$. (Note that we used all edges adjacent with u , all edges adjacent with v , and all edges on the C_n , each edge exactly once. Note also that this is the turning trick applied to the initial P_3 , $u01v$.) Therefore D_n is decomposable into P_3 .