This exam is due Tuesday, March 4, in class. You may consult the text, handouts and homework solutions for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. Complete the table by finding all nonnegative integers m and n for which the graphs below have the following properties:

	cubic	acyclic	Eulerian	Hamiltonian
C_n				
$K_{m,n}$				
K_n				

2. Complete the following: "A graph such that there is at most one path between every two vertices is a ____________."

3. (a) Give an example of a graph in which every 3 vertices are in a common cycle, but which can be disconnected by removing only 2 vertices. *Hint:* A simple example will do.

   (b) *(extra credit)* Give an example of a cubic graph which is not Hamiltonian.

4. Consider a tree with p vertices. An internal vertex is one which has degree ≥ 2. Let I denote the set of internal vertices, and say $|I| = i$ (that is, there are i of them). Prove that
$$\sum_{v \in I} \deg v = p + i - 2$$
(i.e. the sum of the degrees of internal vertices is $p + i - 2$). *Hint:* What are the remaining vertices? How many of them are there?

5. A graph with six vertices has a vertex of degree 1, one of degree 2, one of degree 3, one of degree 4 and one of degree 5. What is the degree of the remaining vertex?

6. How many edges does K_6 have? Is K_6 decomposable into P_2? Is K_6 decomposable into P_5? Prove your answers.

7. Prove the following theorem: A graph is a tree if and only if it is connected and every edge is a bridge. *Hint:* Recall that this requires a two-part proof ("if" and "only if").

8. Prove that the Ramsey number $r(3, 4) = r(K_3, K_4) > 8$ by decomposing the complete graph K_8 into two graphs G and $\overline{G}$ such that G has no K_3 as a subgraph and $\overline{G}$ has no K_4 as a subgraph.

9. The *wheel* W_n is constructed by taking the cycle C_n, then adding a new vertex v (called the *hub* vertex) and edges from v to every vertex on the cycle (think of them as spokes).

   (a) What is the degree sequence of W_n? How many vertices and edges does it have?

   (b) Is W_n connected; Hamiltonian; Eulerian; decomposable into P_2? Prove your answers.

   (c) Decompose W_3 and W_4 into 2 isomorphic subgraphs.

   (d) Prove that W_n is not decomposable into two isomorphic subgraphs for odd $n \geq 7$. *Hint:* Use a proof by contradiction. Consider the larger of the degrees of the hub vertex in the two subgraphs.

   (e) *(extra credit)* Prove that W_5 is not decomposable into 2 isomorphic subgraphs.

10. The *double wheel* D_n is constructed by taking the cycle C_n, then adding two new vertices u and v (the two hubs) and edges from each of them to every vertex on the cycle. There is no edge between u and v. (You can visualize it as two identical cones glued to each other at the base.)

   (a) What is the degree sequence of D_n? How many vertices and edges does it have?

   (b) Is D_n connected; Hamiltonian? For which n is D_n Eulerian? Prove your answers.

   (c) Prove that D_n is decomposable into P_3. *Hint:* Find an initial copy of P_3 and use a turning trick.