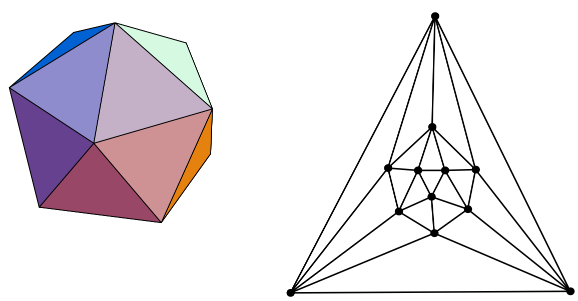
1. (a) The smallest chromatic number for a nonplanar graph is 2. Indeed a graph with $\chi = 1$ cannot have any edge, hence must be planar. However, there are nonplanar graphs with $\chi = 2$, for example, $K_{3,3}$.

   (b) The smallest order (number of vertices) of a planar graph $G$ with $\chi(G) = 4$ is $p = 4$. Indeed, any coloring of $G$ requires at least 4 colors, so $G$ must have at least 4 vertices. On the other hand, the graph $K_4$ is planar, has exactly 4 vertices, and $\chi(K_4) = 4$.

2. (a) Petersen graph is not planar because it contains a subdivision of $K_{3,3}$. To obtain $K_{3,3}$, delete any vertex from the Peterson graph, then in the resulting graph, merge the pairs of edges incident with each vertex of degree 2.

   (b) $C_3 \times C_3$ is not planar because it contains a subdivision of $K_5$ and a subdivision of $K_{3,3}$. For example, if the vertices of each $C_3$ are labeled $1, 2, 3$, use vertices $22, 12, 32, 21, 23$ as the vertices of the subdivision of $K_5$ in $C_3 \times C_3$; or vertices $11, 22, 33; 12, 23, 31$ as the vertices of the subdivision of $K_{3,3}$.

3. To construct the 5-regular maximal planar graph, we will first find its $p$, $q$ and $r$ (numbers of vertices, edges and regions). Since the graph is maximal planar, we have $q = 3p - 6$. Since the graph is 5-regular, its degree sum is $2q = 5p$. Therefore, $5p = 2q = 6p - 12$, so $p = 12$, $q = 30$ and $r = q - p + 2 = 20$. This graph is called *icosahedron*, which can be loosely translated from Greek as "twenty faces" (see Figure 1).



(a) Icosahedron (b) Icosahedron Graph

Figure 1: Icosahedron and Icosahedron Graph

4. To decompose $K_7$ into 2 planar graphs partition edges between the two subgraphs so that one subgraph consists of the outer cycle, all edges incident with one of the vertices inside the cycle, and all edges incident with another vertex outside the cycle (for the total of $7 + 4 + 4 = 15$). This subgraph is maximal planar. The other subgraph will then have $21 - 15 = 6$ edges and can be redrawn as a 5-cycle with a chord inside and 2 isolated vertices.

5. (a) The graph is planar since it can be redrawn as without edge crossings (see Figure 2).

   (b) The graph is not maximal planar since it has $p = 12$ vertices and $q = 26$ edges, and $q < 3p - 6$.

   (c) We must add $3p - 6 - q = 3 \cdot 12 - 6 - 26 = 4$ edges (see dotted edges on Figure 2 for a possible choice) to make the graph maximal planar.

6. (a) Each edge of $C_6$ is adjacent with 2 other edges, so it is not adjacent (hence, may cross) $6 - 1 - 2 = 3$ edges. Thus each edge may cross $\leq 3$ other edges, and each crossing in a simple drawing is an

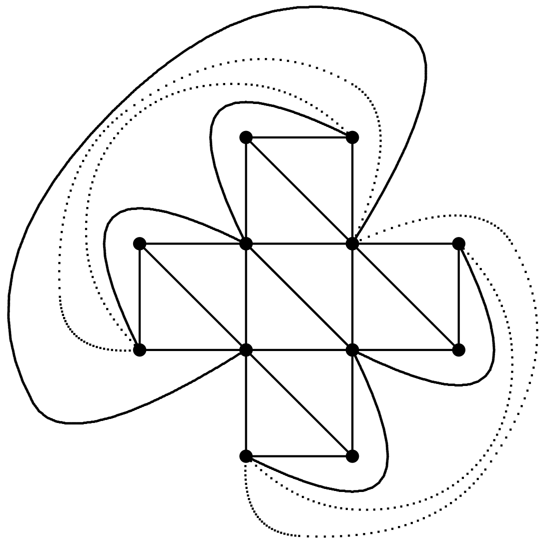


Figure 2: A plane drawing with missing edges (dotted) to make the graph maximal planar

intersection of 2 edges. Let $x$ be the number of crossings, then $2x \leq 3 \cdot 6$, i.e. $x \leq 3 \cdot 6/2 = 9$. Thus, a simple drawing of $C_6$ must have $\leq 9$ crossings.

(b) Figure 3 is a simple drawing of $C_6$ with 9 crossings. Note that no two edges cross more than once, adjacent edges do not cross, and no three edges cross at a single point.

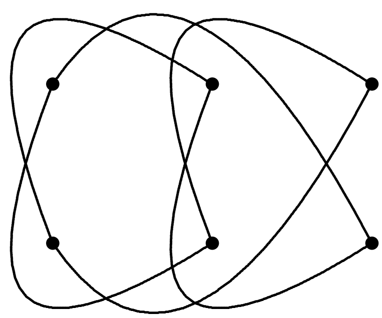


Figure 3: A simple drawing of $C_6$ with 9 edge crossings

7. The edges of the polygons form a normal map, and therefore the dual is maximal planar. Thus, dual graph has $q = 3p - 6$. Each pentagon corresponds to a vertex of degree 5 in the dual, while each hexagon corresponds to a vertex of degree 6 in the dual. Let $p_5$ and $p_6$ be the number of vertices of degrees 5 and 6 respectively (i.e. there are $p_5$ pentagons and $p_6$ hexagons in the normal map). Then $p = p_5 + p_6$ since the dual graph only has vertices of degrees 5 and 6, and the degree sum of the dual is $2q = 5p_5 + 6p_6$. But $2q = 6p - 12$, so $5p_5 + 6p_6 = 6(p_5 + p_6) - 12$, i.e. $p_5 = 12$. Thus, Molly used 12 pentagons each time.