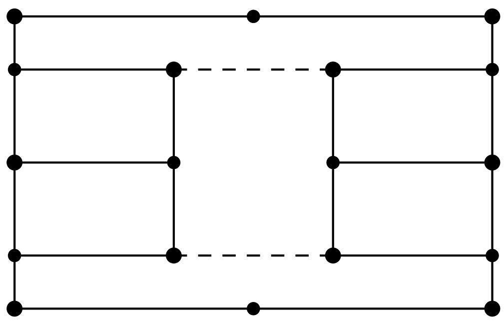
1. The tennis graph is not bipartite because it has odd cycles, for example, the top and bottom C_7 (the long thin boxes). Since these two cycles are disjoint, we need to delete at least 1 edge from each of them. So we need to remove ≥ 2 edges. However, if we remove 2 edges as in the picture below, the resulting graph is bipartite since it has a 2-coloring (this is easy to produce, just start by coloring any vertex red, then color its neighbors blue, then color neighbors of those neighbors red, etc., until all vertices are colored). Thus, the smallest number of edges one needs to delete to obtain a bipartite graph is 2.



2. (a) Since the hub vertex of W_n is adjacent to all vertices of the cycle C_n, it must have a color not used on any vertex of C_n. Thus, we need ≥ 1 more color for a coloring of W_n than for a coloring of C_n, so $\chi(W_n) \geq \chi(C_n) + 1$. On the other hand, given any coloring of C_n, assigning a new color to the hub vertex yields a coloring of W_n, so we need ≤ 1 more color for a coloring of W_n than for a coloring of C_n, so $\chi(W_n) \leq \chi(C_n) + 1$. Therefore, $\chi(W_n) = \chi(C_n) + 1$, i.e. 3 if n is even, and 4 if n is odd.

   (b) Clearly, $\chi(D_n) \geq \chi(W_n)$ since W_n is a subgraph of D_n. On the other hand, the two hub vertices of D_n are not adjacent, hence, can be colored with the same color. Therefore, we do not need any more colors for a coloring of D_n than for a coloring of W_n, so $\chi(D_n) \leq \chi(W_n)$. Thus, $\chi(D_n) = \chi(W_n)$, i.e. 3 if n is even, and 4 if n is odd.

   (c) The dodecahedron graph D has odd cycles (many C_5's at the very least), so $\chi(D) \geq \chi(C_5) = 3$. On the other hand, we can color the vertices of D with 3 colors (check!), so $\chi(D) \leq 3$. Thus, $\chi(D) = 3$.

   (d) Assign colors 0 and 1 to vertices of the n-cube Q_n based on its binary n-tuple label as follows. Color a vertex with color 0 if its label has an even number of 1's, and with color 1 if its label has an odd number of 1's. Since the labels of adjacent vertices differ in exactly one bit, a vertex with an even number of 1's in its label is only adjacent to vertices with an odd number of 1's in its label, and vice versa. Therefore, this 2-coloring is valid, so $\chi(Q_n) \leq 2$. On the other hand, Q_n has edges for $n \geq 1$, so $\chi(Q_n) \geq 2$ for $n \geq 1$. Thus, $\chi(Q_n) = 2$ for $n \geq 1$, and $\chi(Q_0) = 1$.

3. (a) The maximum degree of W_n is $\Delta(W_n) = n$ (the degree of the hub vertex, so $\chi'(W_n) \geq n$. On the other hand, we can color the edges of W_n properly with n colors (remainders $0, 1, 2, \ldots, n-1$ from division by n) as follows. Pick a direction around the cycle. Color the spokes (edges incident with the hub) with colors $0, 1, 2, \ldots, n-1$ consecutively in the chosen direction around the cycle. Now color the cycle edge adjacent to spokes colored k and $k+1$ with color $k+2$ (since we are dealing with remainders modulo n, we have $n \equiv 0 \pmod{n}$, $n+1 \equiv 1 \pmod{n}$, etc.). Thus, if a cycle vertex is incident with a spoke colored k, then it is also incident with cycle edges colored $k+1$ and $k+2$ (check!). Therefore, no adjacent edges have the same color, so this is a proper edge-coloring. Hence, $\chi'(W_n) \leq n$, so $\chi'(W_n) = n$.

   (b) The maximum degree of D_n is $\Delta(D_n) = n$ for $n \geq 4$, while $\Delta(D_3) = 4$. Therefore, $\chi'(D_n) \geq n$ for $n \geq 4$. However, it is easy to produce a proper n-edge-coloring of D_n given the proper n-edge-coloring of W_n above. Namely, color the top W_n subgraph of D_n with n colors as in the previous part, then assign color $k+3 \pmod n$ to the bottom spoke which is adjacent to the top spoke colored k. Then the bottom hub is incident with edges colored $3, 4, 5, \ldots, n-2, n-1, 0, 1, 2$, and each cycle vertex is incident with edges colored $k, k+1, k+2, k+3$ modulo n. Hence, this is a proper n-edge-coloring of D_n for $n \geq 4$, so $\chi'(D_n) \leq n$ for $n \geq 4$. Thus, $\chi'(D_n) = n$ for $n \geq 4$. On the other hand, $\chi'(D_3) \leq 4+1 = 5$, and there is no proper 4-edge-coloring of D_3 (check!), so $\chi'(D_3) > 4$. Therefore, $\chi'(D_3) = 5$.

(c) The dodecahedron graph D is 3-regular, so $\chi'(D) \geq 3$. On the other hand, there is a proper 3-edge-coloring of D, so $\chi'(D) \leq 3$. Thus, $\chi'(D) = 3$.

(d) The n-cube Q_n is n-regular since each vertex has n neighbors (each neighbor's binary label differs from the label of our vertex in exactly one of the n bits). Therefore, $\Delta(Q_n) = n$, so $\chi'(Q_n) \geq n$. Now assign colors $1, 2, \ldots, n$ to edges of the n-cube Q_n based on the binary n-tuple labels of their endpoints as follows. If the labels of the endpoints of an edge differ in position k, assign color k to the edge. Then, every vertex will be incident with 1 edge of each of the n colors, so this n-edge-coloring is proper. Therefore, $\chi'(Q_n) \leq n$, so $\chi'(Q_n) = n$.

4. (a) Consider any 3-edge-coloring of K_{17} (say, with colors red, blue and green). Each vertex of K_{17} has degree $17-1 = 16$, so there is a color (say, green) which is assigned to at least $16/3 > 5$ vertices, i.e. to at least 6 vertices. Consider the the other endpoints of 6 green edges, and the edges between those endpoints (which form a K_6). If any edge between them is also colored green, then we have a green triangle. If not, then the edges of that K_6 are colored red or blue. But we know that $r(3,3) = 6$, so any red-and-blue edge-coloring of K_6 there must contain a red triangle or a blue triangle. Thus, in any case there must be a monochromatic triangle (K_3 with all edges of the same color).

(b) Consider a 3-edge-coloring of K_{16}. As in part (a), if the red degree or the blue degree or the green degree of any vertex is at least 6, then there is a monochromatic triangle. Therefore, to avoid a monochromatic triangle, each vertex must be incident to ≤ 5 edges of each color. However, each vertex of K_{16} has degree $16-1 = 15 = 5+5+5$, so no vertex can be incident with < 5 edges of any color. Therefore, each vertex must be incident with exactly 5 edges of each color, so the subgraph of each color must be a 5-regular spanning subgraph of K_{16}, i.e. a 5-factor of K_{16}.

5. Let G be a graph such that for every pair of vertices v and w in G, $\chi(G-v-w) = \chi(G)-2$. Color the subgraph $G-v-w$ of G with $\chi(G-v-w) = \chi(G)-2$ colors. Now if the vertices v and w are not adjacent we can color them with the same color and produce a coloring of G with $\chi(G)-2+1 = \chi(G)-1 < \chi(G)$ colors. But that is impossible by definition of $\chi(G)$. Therefore, any two vertices v and w of G must be adjacent, so G must be complete.

6. (a) Suppose we colored G with $\chi(G)$ colors and H with $\chi(H)$ colors. Let (g,h) be a vertex in $G \times H$, and suppose g gets color i in G and h gets color j in H. Then assign color (i,j) to vertex (g,h). Check (using the definition of $G \times H$) that this is a valid coloring which uses $\chi(G)\chi(H)$ colors. Therefore, $\chi(G \times H) \leq \chi(G)\chi(H)$.

(b) Without loss of generality, suppose that $m \leq n$. Now color vertex (g,h) with color $i+j \pmod m$ (i.e. the remainder from division of $i+j$ by m). Suppose (g_1,h_1) and (g_2,h_2) get the same color in this coloring. Let g_1 get color i_1 and g_2 get color i_2 in G, and let h_1 get color j_1 and h_2 get color j_2 in H, then $i_1+j_1 \equiv i_2+j_2 \pmod n$. If (g_1,h_1) and (g_2,h_2) are adjacent in $G \times H$, then either $g_1 = g_2$ and h_1 and h_2 are adjacent in H or $h_1 = h_2$ and g_1 and g_2 are adjacent in G. Without loss of generality, assume that $g_1 = g_2$. Then $i_1 = i_2$, so $j_1 \equiv j_2 \pmod n$. But $0 \leq j_1, j_2 \leq n-1$, so we must have $j_1 = j_2$. But h_1 is adjacent to h_2 in H, so they cannot get the same color. Contradiction. Hence, (g_1,h_1) and (g_2,h_2) are not adjacent in $G \times H$ if they receive the same color. Hence our coloring is valid, so $\chi(G \times H) \leq \max(\chi(G), \chi(H))$. On the other hand, $G \times H$ contains both G and H as subgraphs, so $\chi(G \times H) \geq \max(\chi(G), \chi(H))$. Thus, $\chi(G \times H) = \max(\chi(G), \chi(H))$.

7. We will show that if the longest cycle is less than m then we can make it even longer, thus producing a contradiction, which will imply our result. Suppose that our graph satisfies the degree condition, but

its longest path has length $k \leq m-1$. Let u and v be the endpoints of this path (call it P_k). (Recall that P_k has k edges and $k+1$ vertices.) Now u and v cannot be adjacent to any vertex outside of P_k, otherwise we would be able to extend P_k even more and it would not be a longest path in G. Hence, we have that all the neighbors of u and v lie on P_k, and $\deg(u) + \deg(v) \geq m \geq k+1$. This is exactly the situation we had in the proof of Ore's theorem, so we can use the same trick to construct a cycle C_{k+1} out of P_k. Hence, G contains C_{k+1} as a subgraph. But $k+1 \leq m < p$, so G has vertices outside of C_{k+1}. Furthermore, G is connected, so at least one of the outside vertices (say, w) is adjacent to at least one vertex on C_{k+1} (say, x). Let y a vertex adjacent to x on C_{k+1}. Adjoin vertex w and edge wx to C_{k+1} and erase edge xy to get a path P_{k+1}. We have thus found a path of length $k+1$ in G. This contradicts the maximality of P_k, therefore the longest path in G must have length $\geq m$, so G contains a path of length m.