1. Is the tennis graph of Homework 2 bipartite? Prove your answer. If no, what is the smallest number of edges you need to remove to obtain a bipartite graph? Prove your answer.

2. Determine the chromatic numbers $\chi$ of the following graphs:
   (a) the wheel $W_n$;
   (b) the double wheel $D_n$;
   (c) the dodecahedron graph $D$;
   (d) the $n$-cube $Q_n$. *Hint:* Assign a color to a vertex based on its binary $n$-tuple label.

3. Determine the edge-chromatic numbers $\chi'$ of the following graphs:
   (a) the wheel $W_n$;
   (b) the double wheel $D_n$;
   (c) the dodecahedron graph $D$;
   (d) the $n$-cube $Q_n$. *Hint:* Assign a color to an edge based on the labels of its endpoints.

4. (a) Prove that any edge-coloring of $K_{17}$ with 3 colors contains a monochromatic $K_3$ (a triangle with all edges of the same color). [*Note:* This proves that $r(3,3,3) \leq 17$.] *Hint:* Use the fact that $r(3,3) = 6$ and $16/3 > 5$.
   (b) In fact, it is known that $r(3,3,3) = 17$. In other words, in addition to part (a), there is a 3-edge-coloring of $K_{16}$ that does not contain a monochromatic triangle. Prove that the subgraph of each color in such a coloring must be a 5-factor of $K_{16}$.
   (c) *(extra credit)* Find a 3-edge-coloring of $K_{16}$ as described in the previous part. *Warning:* This is *really* hard, but I'd love to see you succeed. For this part, you may use any outside sources, however you must understand the solution you present (and I will check!). You may turn in this part anytime before the final exam is due. A bonus extra credit will be given for a solution which has a nice visualization (e.g., each color subgraph is centrally symmetric, or the three subgraphs may be obtained from one another by rotation, etc.).

5. Prove that if $\chi(G - v - w) = \chi(G) - 2$ for every pair of vertices $v$ and $w$ in a graph $G$, then $G$ is complete. *Hint:* Prove the contrapositive: If $G$ is not complete, then $\chi(G - v - w) \geq \chi(G) - 1$ for some pair of vertices $v$ and $w$ in $G$.

6. Consider the product graph $G \times H$ of two graphs $G$ and $H$.
   (a) Prove that $\chi(G \times H) \leq \chi(G)\chi(H)$. *Hint:* Color each vertex $(g, h)$ in $G \times H$ based on the colors of vertex $g$ in $G$ and vertex $h$ in $H$.

(b) *(extra credit)* Prove that in fact $\chi(G \times H) = \max(\chi(G), \chi(H))$. *Hint:* For colors, use remainders modulo $\max(\chi(G), \chi(H))$.

7. *(extra credit)* Let $G$ be a connected graph on $p$ vertices and let $m < p$. Prove that if for all distinct nonadjacent vertices $u$ and $v$ we have $\deg(u) + \deg(v) \geq m$, then $G$ contains a path of length $m$. *Hint:* Consider the longest path in $G$ and its end vertices. Use an argument similar to that in the proof of the Ore's theorem.

8. *(extra credit)* Consider the Hamilton cycle of $Q_n$ (Hamilton path for $Q_1$) constructed recursively as in the Sketch of Solutions to Homework 2. For example:

   - empty set $\emptyset$ for $Q_0$,
   - $0 - 1$ for $Q_1$,
   - $00 - 01 - 11 - 10(-00)$ for $Q_2$,
   - $000 - 001 - 011 - 010 - 110 - 111 - 101 - 100(-000)$ for $Q_3$,
   - $0000 - 0001 - 0011 - 0010 - 0110 - 0111 - 0101 - 0100 - 1100 - 1101 - 1111 - 1110 - 1010 - 1011 - 1001 - 1000(-0000)$ for $Q_4$.

   Clearly, to find the next vertex in this Hamilton cycle of $Q_n$ (also called the Gray code) you only need to change one coordinate every time. Find an algorithm which take as input a label of a vertex in $Q_n$ (i.e. a binary string of length $n$) and output the next vertex along the Hamilton cycle constructed as above (i.e. the next element in the Gray code). Give a proof that your algorithm indeed always finds the next element in the Gray code. *Note:* Your algorithm should *not* construct all Hamilton cycles recursively. In fact, the number of steps it should take must be a linear function of $n$ (the length of the input string). Don't just write code and let me decipher it. Explain in words how your algorithm works. This problem may be turned in anytime before the final exam is due.