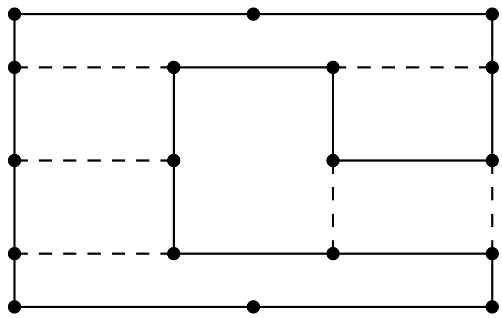
1. (a) Yes, the dodecahedron graph is Hamiltonian. It is not difficult to find a Hamilton cycle.

   (b) No, the graph G on Figure 3.21 is not Hamiltonian. Too see this, recall that a Hamilton cycle H of G, if there is one, must contain all vertices of G; in particular, all the outermost vertices. The only way to have each such vertex in a cycle is to include both edges incident with it. But then we obtain a cycle C which contains all vertices except the middle one and uses all the outermost edges. Now C must be a subgraph of any Hamilton cycle. But one cycle is a subgraph of another if and only if they are the same, so $H = C$. This is a contradiction, since H must include the middle vertex, which C does not.

2. (a) The tennis graph has a Hamilton cycle (one such cycle is drawn below). Clearly, given any edge of the tennis graph, we can reflect the Hamilton cycle below horizontally and/or vertically so that it contains that edge.



A Hamilton cycle of the tennis graph.

   (b) If the groundskeeper for such a tennis court is using his machine to put the chalk lines on the tennis court, what is the minimum number of times he must interrupt his machine?

   The tennis graph has 12 vertices of odd degree (all vertices except the top and bottom row), so it cannot be decomposed into less than $12/2 = 6$ trails. Therefore, the groundskeeper will have to interrupt his machine at least 6 times so as to paint each edge exactly once. Note that he should not start at the corners as these are vertices of even degree!

3. Using the street map in Problem 12, Section 3.1, construct a multigraph G with intersections as vertices and edges as segments of sides of streets between adjacent intersections. Note that every pair of adjacent vertices is connected by 2 edges, one for each side of the street. Clearly, each vertex of G has an even degree (twice the number of streets incident with the corresponding intersection). Also, G is obviously connected. Therefore, G must be Eulerian. In other words, G has a circuit that uses each edge exactly once. If the letter carrier follows an Eulerian circuit of G, she will use each edge exactly once, in other words, she will walk on each side of every street exactly once.

   Furthermore, note that our proof did not use the shape of the actual street map of Problem 12, and hence applies to *any* (connected) street map!

4. Let v be any vertex of G and let c denote the number of cycles through v. For any edge e incident with v, let c_e denote the number of cycles through e. Sum up, over all cycles through v, the numbers of edges incident with v which lie on cycles through v. Since every cycle through v uses 2 edges incident with v, this sum is

$$\underbrace{2 + 2 + \cdots + 2}_{c} = 2c.$$

   On the other hand, each edge e incident with v is counted in this sum as many times as there are cycles which contain e, i.e. c_e times. Therefore, our sum is also

$$\sum_{e \text{ incident with } v} c_e.$$

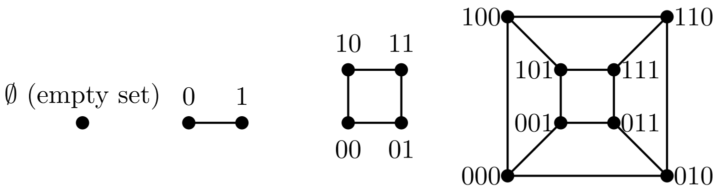
But this the same sum, so

$$\sum_{e \text{ incident with } v} c_e = 2c.$$

Since all summands c_e are odd and the sum $2c$ is even, there must be an even number summands. So v is incident with an even number of edges, i.e. has even degree. Hence, all vertices of G have even degree, so G contains an Eulerian circuit.

Note: The converse is also true. In other words, if a graph is Eulerian, then each of its edges lies on an odd number of cycles.

5. (a) $Q_0 = K_1$, $Q_1 = K_2$, $Q_2 = C_4$, Q_3 is indeed the same graph as in Problem 7, Homework 1. To see that, label the vertices of the graphs on the right appropriately as below or in Figure 6.11 on page 137.

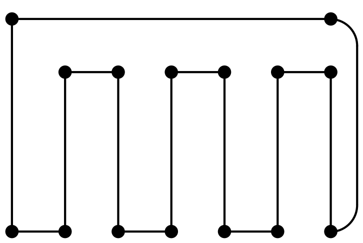


(b) A label of every vertex in Q_n has n coordinates, and any neighbor is produced by changing exactly 1 of these coordinates. Hence, every vertex has n neighbors, i.e. degree n. Therefore, Q_n is n-regular. Thus, Q_n is Eulerian if and only if n is even.

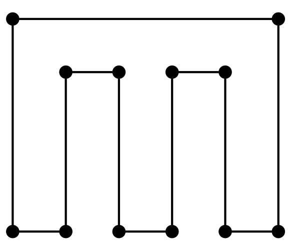
(c) The proof here is by induction. Obviously, $Q_2 = C_4$ is Hamiltonian. Assume that Q_{n-1} is Hamiltonian. Take a Hamilton cycle C_p of Q_{n-1} (where $p = 2^{n-1}$ is the number of vertices of Q_{n-1}). Travel along this cycle among those vertices in Q_n whose first coordinate is 0, say from $(0,0,0,0,\dots,0)$ to $(0,1,0,0,\dots,0)$, then to $(1,1,0,0,\dots,0)$ along an edge of Q_n, then along the same cycle in reverse order among those vertices which have first coordinate 1, i.e. from $(1,1,0,0,\dots,0)$ to $(1,0,0,0,\dots,0)$, then back to $(0,0,0,0,\dots,0)$. For example, a Hamilton cycle in Q_3 produced in this way from a Hamilton cycle in Q_2, say $00-01-11-10-00$, is $000-001-011-010-110-111-101-100-000$. (Such a list of n-bit binary strings, where consecutive strings differ in exactly one bit, is called the *Gray code*. This is the most efficient way to produce a list of all n-bit binary strings. Also see Section 6.3 for an application of this problem to solving the Tower of Hanoi puzzle.)

6. (a) Let v be a vertex of Q_n with coordinates $v_1 v_2 \dots v_n$. Then rewrite it as follows: $(v_1, v_2 \dots v_n)$. Clearly, the first element of that pair gives a coordinate of a vertex in $Q_1 = K_2$ and the second element gives coordinates of a vertex in Q_{n-1}. Similarly, given a pair of vertices in $Q_1 \times Q_{n-1}$ (given by their coordinates), say $(v_1, v_2 \dots v_n)$ we can obtain a vertex of Q_n given by $v_1 v_2 \dots v_n$. Two vertices in Q_n are adjacent if and only if they differ in the first coordinate but their other coordinates are the same, or they have the same first coordinate but differ in another coordinate. That, together with the above mapping, fits exactly into the definition of the product $Q_1 \times Q_{n-1}$.

(b) Suppose G has m vertices and H has n vertices. Take a Hamilton cycle C_m of G and a Hamilton path P_{n-1} of H (the rest of the edges are irrelevant here). The picture below shows the Hamilton cycle for $C_m \times P_{n-1}$ (which you can visualize as a rectangular grid on a cylinder).



n odd



n even