Sketch of Solutions to Assignment 1

1. (a) not graphic (sum of degrees is odd). (b) graphic. (c) graphic. (d) graphic.

2. Let G be any graph on p vertices, and let Δ and δ be the maximum and minimum degree of G, respectively. Then $\Delta \leq p-1$ and $\delta \geq 0$. Suppose the degree sequence of G has all degrees different. Then it contains p integers (as many as there are vertices) between 0 and $p-1$. But there are exactly p integers between 0 and $p-1$. But $\Delta = p-1$ and $\delta = 0$. Hence, our graph must contain a vertex of degree 0, so a vertex of degree Δ can be adjacent to at most $p-2$ vertices. In other words, $\Delta \leq p-2$. Contradiction. Thus, the degree sequence of any graph must contain a repeated element.

3. Graph G_3 is the only one not isomorphic to the cube graph Q_3, since it contains an odd cycle C_5 as a subgraph, which Q_3 does not (see below). To see that the other two graphs are isomorphic to Q_3, number the vertices correspondingly. To see that Q_3 does not contain C_5 note that we can color the vertices of Q_3 with two colors so that no two vertices of the same color are adjacent. (This is especially easy to do with Q_3 drawn as G_1 if we color the top vertices with one color and the bottom vertices with the other color.) Obviously, we cannot do the same with C_5, so C_5 cannot be a subgraph of Q_3.

4. There are 11 non-isomorphic trees on 7 vertices. To see this, classify the trees according the number of leaves. Then classify them further according to their degree sequence. Any tree on 7 vertices has $7-1=6$ edges, so the sum of all degrees of its vertices must be $2 \cdot (7-1) = 12$. Any tree must also have at least two leaves, i.e. two vertices of degree 1. Note that nonleaves in a connected graph such as a tree have degrees ≥ 2.

   Case 1: 6 leaves and 1 nonleaf with degree $12-6=6$. Hence, the degree sequence is $6,1,1,1,1,1,1$, which yields only the *star* tree S_6.

   Case 2: 5 leaves and 2 nonleaves whose degrees sum to $12-5=7$. This yields two degree sequences: $5,2,1,1,1,1,1$ and $4,3,1,1,1,1,1$. Each of these sequences yields a unique tree.

   Case 3: 4 leaves and 3 nonleaves whose degrees sum to $12-4=8$. This yields degree sequences $4,2,2,1,1,1,1$ and $3,3,2,1,1,1,1$. The sequence $4,2,2,1,1,1,1$ yields two non-isomorphic trees: the vertex of degree 4 is adjacent to a leaf in one and non-adjacent to leaves in the other. The sequence $3,3,2,1,1,1,1$ also yields two non-isomorphic trees: the two vertices of degree 3 are adjacent in one and non-adjacent in the other.

   Case 4: 3 leaves and 4 nonleaves whose degrees sum to $12-3=9$. This yields a unique degree sequence: $3,2,2,2,1,1,1$, which yields three non-isomorphic trees: the vertex of degree 3 is adjacent to 2 leaves in one, 1 leaf in the other, and no leaf in the third tree.

   Case 5: 2 leaves and 5 nonleaves whose degrees sum to $12-2=10$. This yields a unique degree sequence: $2,2,2,2,2,1,1$, which yields the path P_6 (6 edges).

5. (a) Any graph must have even number of odd vertices. Since all vertices in G have odd degrees, G must have an even number p of vertices. Since G is a tree, the number of edges in G, $q = p-1$, hence q is odd.

   (b) K_4 has all 4 vertices of degree 3 and $(3 \cdot 4)/2 = 6$ edges.

6. We will show that if every vertex in a graph G has degree at least $(p-1)/2$, then any two vertices are either adjacent or have a common neighbor, hence G is connected. Suppose neither is true for some vertices u and v of G. Then the number of vertices in G is at least $1+(p-1)/2+1+(p-1)/2 = p+1$ (u and its neighbors plus v and its neighbors). But $p+1 > p$, hence we have a contradiction, so our assertion above is true.

7. (a) $K_{3,3}$ (why?). (b) Three copies of K_3 joined at one vertex (why?).