This exam is due Wednesday, May 7, by 5pm, in my office (Carver 456). You may consult the text, handouts and homework solutions for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK. Mere oracular answers will get little, if any, credit.

---

1. Complete the table by finding all nonnegative integers m and n for which the graphs below have the following properties:

	C_n	$K_{m,n}$	K_n	W_n	D_n
planar					
maximal planar					

2. A bipartite graph is called *balanced* if every 2-coloring of the vertices uses each color the same number of times.

   (a) Prove that a regular bipartite graph of positive degree must be balanced. *Hint:* Take any red-and-blue coloring of the graph. Count the edges as starting at red vertices and as starting at the blue vertices. *Caution:* This bipartite graph does not have to be complete bipartite.

   (b) Prove that a Hamiltonian bipartite graph on ≥ 2 vertices must be balanced. *Hint:* Take any red-and-blue coloring of the graph and consider the Hamilton cycle.

3. For an integer $n \geq 2$, the graph M_n is defined by taking a cycle C_{2n} on $2n$ vertices $1, 2, \ldots, 2n$ and joining each pair of opposite vertices (j and $n + j$ for $j = 1, 2, \ldots, n$) on the cycle. (So M_n is 3-regular.)

   (a) Find $\chi(M_n)$. Prove your answer. *Hint:* The answer depends on whether n is odd or even. There is also a small special case.

   (b) Find $\chi'(M_n)$. Prove your answer.

   (c) Prove that M_n is not planar for $n \geq 3$. *Hint:* First consider M_3 (which known graph is it?), then use Kuratowski's theorem.

   (d) Prove that the crossing number $cr(M_n) = 1$ for $n \geq 3$. *Hint:* Recall that $cr(M_n)$ is the least number of crossings any simple drawing of M_n in the plane must have. To find it, redraw M_n in a completely different way. Start by uncrossing as many edges as possible.

4. (a) Prove that if n is odd, then D_n is not decomposable into P_6. *Hint:* Count edges.

(b) Prove that if $n \geq 6$ is even, then D_n is decomposable into P_6. *Hint:* Find initial P_6 and use the turning method.

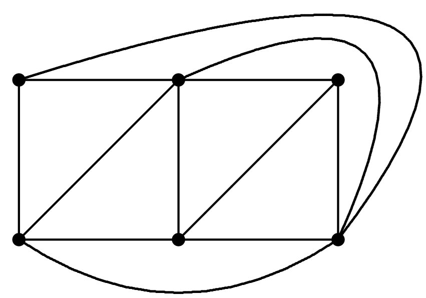
(c) Prove that D_n is never decomposable into P_7. *Hint:* Counting argument. How many copies of P_7 would D_n have to contain? How many of those edges would be adjacent to each hub? Find an upper bound and derive a contradiction.

5. Prove that the degree sequence $(n, n, n-1, n-1, , n-2, n-2, \ldots, 2, 2, 1, 1)$ is graphic. *Hint:* Construct a graph with this degree sequence as follows. Take $2n$ vertices $u_1, \ldots, u_n; v_1, \ldots, v_n$ and decide whether or not to join each pair u_i and v_j by an edge based on whether or not the pair of indices i and j satisfies a certain condition to be discovered by you. Then prove that in your graph $\deg(u_i) = \deg(v_i) = i$ for each $i = 1, 2, \ldots, n$.

6. Find the dual graph for the icosahedron graph of Problem 3, Homework 4. Identify it as some previously known graph. Prove your answer.

7. Find the labeled tree associated by the Prüfer method with the sequence $(4, 4, 1, 3, 3, 1)$.

8. Find another plane drawing of the graph below where all edges are straight lines. (Label the vertices on both drawings so that the isomorphism is apparent.) (*extra credit* for a nice, symmetric drawing)



9. Connect A to a, B to b, C to c, and D to d in Figure 1 by edges that are inside the rectangle and do not cross each other.

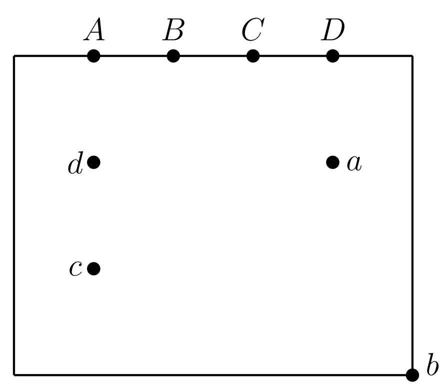


Figure 1: Dudeney's Puzzle

10. *(extra credit)* How many letters does the answer to this question have? *(this does not refer to any previous problem)*