1 Decomposition of Graphs

Definition 1.1 A graph G is *decomposable* into $H_1, H_2, \ldots, H_k$ if G has subgraphs $H_1, H_2, \ldots, H_k$ such that

1. each edge of G belongs to one of the H_i's for some $i = 1, 2, \ldots, k$; and
2. if $i \neq j$, then H_i and H_j have no edges in common.

Definition 1.2 Let G and H be two graphs. A graph G is *decomposable into H's* if each of the H_i's in the definition above is isomorphic to H.

2 Decomposition into Hamilton cycles and Hamilton paths

Example 2.1 K_5 is decomposable into 2 C_5's (the rim 123451 and the star 135241).

Example 2.2 K_7 is decomposable into 3 C_7's as follows: label vertices clockwise around the circle and always go to the next vertex in the first copy of C_7, skip 1 vertex in the second copy of C_7, and skip 2 vertices in the third copy of C_7 (12345671, 13572461, 14736251).

We can't extend this to K_9 (specifically, we get stuck at 1471 skipping 2 vertices). As it turns out, K_9 is decomposable into C_9 anyway, we just need a more clever trick called the *turning trick*.

Theorem 2.3 *For any $n \geq 1$, K_{2n+1} is decomposable into n Hamilton cycles C_{2n+1}.*

Proof. Label the vertices $\infty, 0, 1, 2, \ldots, 2n-1$, and form the cycles as follows:

$$\begin{array}{ll} (C_1) & \infty, 0, 2n-1, 1, 2n-2, 2, 2n-3, \ldots, n-1, n, \infty \\ (C_2) & \infty, 1, 0, 2, 2n-1, 3, 2n-2, \ldots, n, n+1, \infty \\ (C_3) & \infty, 2, 1, 3, 0, 4, 2n-1, \ldots, n+1, n+2, \infty \\ \vdots & \\ (C_n) & \infty, n-1, n-2, n, n-3, n+1, n-4, \ldots, 2n-2, 2n-1, \infty \end{array}$$

(so each time we add 1 to every label and find the remainder modulo $2n$). For example, label vertices of K_9 with $\infty, 0, 1, 2, 3, 4, 5, 6, 7$ (so $n = 4$) and decompose it into $\infty 07162534\infty$,

$\infty 10273645 \infty$, $\infty 21304756 \infty$, $\infty 32415067 \infty$. (You can visualize this by putting vertices 0 through $2n-1$ clockwise along a circle with ∞ outside with circle.) □

Corollary 2.4 *For any $n \geq 1$, K_{2n} is decomposable into n Hamilton paths P_{2n-1}.*

PROOF. Take the composition of K_{2n+1} of Theorem 2.3 and delete the ∞ vertex. K_{2n+1} will become K_{2n}, while each Hamilton cycle C_{2n+1} of K_{2n+1} will become a Hamilton path P_{2n-1} of K_{2n}. □

On the other hand,

Theorem 2.5 *K_{2n} is not decomposable into Hamilton cycles.*

This time the proof is purely numerical: the numbers don't add up.

PROOF. Each vertex of K_{2n} has degree $2n-1$. Let q be the number of edges of K_{2n}, then $2q = 2n(2n-1)$, i.e. $q = n(2n-1)$. Suppose K_{2n} is decomposable into Hamilton cycles C_{2n}. Each C_{2n} has $2n$ edges, so the number of such cycles in the decomposition must be

$$\frac{q}{2n} = \frac{n(2n-1)}{2n} = n - \frac{1}{2},$$

which is never an integer. Hence, this is impossible, so K_{2n} is not decomposable into Hamilton cycles. □

3 Decomposition into paths

Example 3.1 K_4 is decomposable into P_2's and into P_3's, but not into P_4's.

Example 3.2 Q_3 is decomposable into P_2's, but not into P_4's.

Theorem 3.3 *A connected graph on q edges is decomposable into P_2's if and only if q is even.*

To prove this theorem, we will introduce an auxiliary definition.

Definition 3.4 A *partial decomposition into P_2's* is a coloring of some of the edges such that each color is used twice and edges with the same color are adjacent. (In other words, edges of each color from a P_2.)

We will now prove the above theorem.

PROOF. ($\Longrightarrow$) Obviously, if G is decomposable into P_2's then the number of edges of G is 2 times the number of copies of P_2 in the decomposition, which is even.

($\Longleftarrow$) Now suppose G is connected and has an even number of edges q. We will prove by contradiction that G is decomposable into P_2's.

Suppose G is not decomposable into P_2's. Take a maximal partial decomposition of G into P_2, in the following sense:

1. it uses the maximum possible number of colors; and
2. the distance (i.e. the number of edges on the shortest path) between the nearest pair of uncolored edges is as small as possible.

Then no two uncolored edges of G can be adjacent (or we can color another copy of P_2, which contradicts maximality of G). Let xy and zt be the closest pair of uncolored edges. Take the shortest path from xy to zt. Let m be the first vertex on this path from xy to zt. Say m is adjacent to y (e.g. $xym \dots zt$ is the shortest path from xy to zt). Then ym is a colored edge. Now we have two cases.

1. ym is in a P_2 with some edge ma. Replace yma by xym in the partial decomposition. Then the newly uncolored edge ma is closer to zt than xy.
2. ym is in a P_2 with some edge ya. Replace aym by ayx in the partial decomposition. Then the newly uncolored edge ym is closer to zt than xy.

Thus, in either case we have a partial decomposition with the same number of colors with the distance between the closest pair of uncolored edges is less than what we originally had. This is a contradiction. Therefore, our assumption is false, so G is decomposable into P_2. □

Informally speaking, what we do in the above proof is bring the closest pair of uncolored edges closer and closer together until they're adjacent, so we can color them and make a new copy of P_2. We repeat this process until there are no uncolored edges left.

4 Regular graphs and decomposition

Definition 4.1 A graph G is called *r-regular* if every vertex of G has degree r.

Example 4.2 A 0-regular graph has only isolated vertices (i.e. it's a disjoint union of K_1's). A 1-regular graph (also called a *matching*) is a disjoint union of K_2's, so it must have an even number of vertices. A 2-regular graph is a disjoint union of cycles. A 3-regular graph is called *cubic*. K_p is $(p-1)$-regular.

Definition 4.3 An *r-factor* of a graph G is an r-regular spanning subgraph of G.

Theorem 4.4 *For any $n \geq 1$, K_{2n} is decomposable into $2n-1$ 1-factors.*

Proof. Label vertices $\infty, 0, 1, 2, \ldots, 2n-2$. Start with a 1-factor

$$\infty, 0; \quad 1, 2n-2; \quad 2, 2n-3; \quad \ldots \quad n-1, n$$

use the turning trick to obtain the following decomposition:

$$\begin{array}{l} \infty, 0; \quad 1, 2n-2; \quad 2, 2n-3; \quad \ldots \quad n-1, n \\ \infty, 1; \quad 2, 0; \quad 3, 2n-2; \quad \ldots \quad n, n+1 \\ \vdots \\ \infty, 2n-2; \quad 0, 2n-3; \quad 1, 2n-4; \quad \ldots \quad n-2, n-1. \end{array}$$

□

Theorem 4.5 *A cubic graph has no decomposition into P_4's.*

Again, we will give a counting argument in a proof by contradiction. Its basic idea is that we need more vertices than we have for a decomposition into P_4's.

Proof. Let G be a cubic graph with p vertices and q edges. Each vertex has degree 3, so $3p = 2q$. Hence, $3p$ is even, so p is even. Let $p = 2m$ for some integer m. Then $q = 3m$. If G is decomposable into P_4, then $q = 4k$, where k is number of P_4's in the decomposition. Hence, $3m = 4k$, so 4 divides $3m$ and there are $3m/4$ copies of P_4 in the decomposition.

Now consider the interior vertices (not leaves) of the P_4's. Each vertex of G can be an interior vertex of ≤ 1 copy of P_4, otherwise it would have degree $\geq 2+2 = 4 > 3$. There are $3m/4$ copies of P_4 in the decomposition, and we need at least 3 interior vertices for each of them. Thus, we need at least a total of $3 \times 3m/4 = 9m/4 > 2m$ vertices. But we only have $2m$ vertices. This is a contradiction, so G is not decomposable into P_4's. □

Corollary 4.6 *A cubic graph is not decomposable into P_n's for any $n \geq 4$.*

The proof is very similar.

Theorem 4.7 *A cubic graph with a bridge is not decomposable into three 1-factors.*

Theorem 4.8 (Petersen) *A cubic graph without a bridge is decomposable into three 1-factors.*

Corollary 4.9 *A cubic graph without a bridge is decomposable into P_3's.*