# Counting

Math 314

May 4, 2003

## 1 Cayley's Spanning Tree Formula with Prüfer's Proof

Recall that a spanning tree of a graph $G$ is a subgraph of $G$ that is a tree containing all vertices of $G$. In 1889, Cayley proved the following theorem.

**Theorem 1.1 (Cayley)** *The number of spanning trees in the complete graph $K_n$ is $n^{n-2}$.*

**Remark 1.2** Note that we are not counting nonisomorphic trees here (i.e. different unlabeled trees as in HW 1), but all labeled trees on $n$ vertices (i.e. trees with vertices labeled by $1, 2, 3, \ldots, n$).

There are many proofs of this elegant result, and one of the simplest and equally elegant of them is due to Prüfer. Before we begin the proof, we make the following observation.

**Fact 1.3** *For $n \geq 2$, $n^{n-2}$ is the number of different ordered sequences $(b_1, b_2, \ldots, b_{n-2})$ of $n-2$ elements from the set $\{1, 2, \ldots, n\}$ with repeated elements allowed.*

Indeed, there are $n$ ways to choose each element $b_i$ in the sequence for $i = 1, 2, \ldots, n-2$, so the total number of ways to choose the whole sequence is $\underbrace{n \cdot n \cdot n \cdot \ \cdots \ \cdot n}_{n-2} = n^{n-2}$.

Proof. The idea of Prüfer's proof is to find a bijection (a one-to-one correspondence) between the set of spanning trees of $K_n$ and the set of sequences of length $n-2$ whose elements are from the set $\{1, 2, \ldots, n\}$. Since there is a unique spanning tree that corresponds to each sequence, and a unique sequence that corresponds to each tree, there must the same number of spanning trees and sequences.

We will present two algorithms, one that produces a sequence given a labeled tree (we will call it the associated sequence) and one that produces a labeled tree given a sequence (we will call it the associated tree), and then show why they are inverse to each other (i.e. one "undoes" the other). Again, note that this does not apply to $n = 0, 1$.

*Algorithm 1.*

1. Initialize the sequence as the empty set.
2. If the tree has $> 2$ vertices, find the leaf with the lowest label. If the tree has 2 vertices, stop.
3. Determine the label of the (unique) neighbor of that leaf and append that label to the sequence.
4. Delete the leaf with the lowest label from the tree and go to Step 2.

Note that we end up deleting all but two vertices from our tree, i.e. $n-2$ vertices, so our sequence indeed has $n-2$ labels. Some of the values in the sequence may be repeated. Also note that leaf labels do not occur in the associated sequence, and any internal vertex $v$ will occur in the sequence every time an adjacent vertex is deleted, and until our original vertex will itself become a leaf, i.e. $\deg(v)-1$ times. Thus, each vertex $v$ of a labeled tree (including leaves) occurs $\deg(v)-1$ times in the associated sequence.

*Algorithm 2.*

1. Add 2 to the length of the sequence to find the number of vertices, $n$.
2. Initialize the set $V$ as $\{1, 2, \ldots, n\}$ and $S$ as our sequence.
3. Initialize the graph as $n$ isolated vertices labeled $1, 2, 3, \ldots, n$.
4. If $S$ is nonempty, find the smallest element $i$ of the set $V$ that does *not* occur in the sequence. If $S$ is empty, connect the 2 elements in $V$ and stop.
5. Connect vertex $i$ to the vertex labeled with the first (leftmost) element of the sequence.
6. Let $V \leftarrow V - \{i\}$ (i.e. delete element $i$ from the set $V$), delete the first (leftmost) element from $S$, and go to Step 4.

It easy to see that the resulting graph is a tree since one of the endpoints of each new edge is a previously isolated vertex. Thus, no edge we add in Algorithm 2 can create a cycle. Also, the resulting acyclic graph on $n$ vertices has $(n-2)+1 = n-1$ edges, so it must be a tree.

To see that Algorithms 1 and 2 indeed describe inverse maps, note that the first loops of each algorithm are inverse to each other (i.e. "undo" each other), as are the second loops, the third loops, etc. $\square$

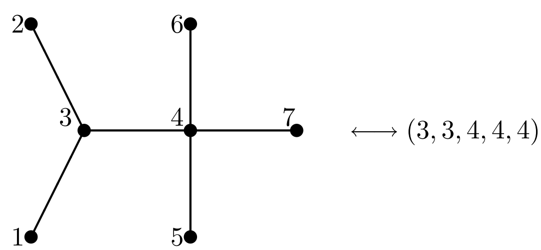


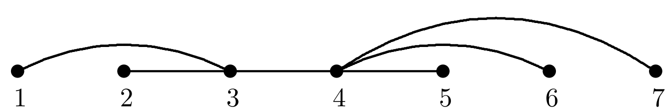

Figure 1: Example of the Prüfer's bijection between labeled trees on $n$ vertices and sequences of $n-2$ elements from $\{1, 2, \ldots, n\}$ with repetitions allowed. In this example, $n = 7$.

**Example 1.4** Let us find the sequence associated by the Prüfer's method to the tree in Figure 1.

1. The smallest leaf is 1, its neighbor is 3. Append 3 to the sequence, delete leaf 1.
2. The smallest leaf is 2, its neighbor is 3. Append 3 to the sequence, delete leaf 2. (Now 3 is a leaf.)
3. The smallest leaf is 3, its neighbor is 4. Append 4 to the sequence, delete leaf 3.
4. The smallest leaf is 5, its neighbor is 4. Append 4 to the sequence, delete leaf 5.
5. The smallest leaf is 6, its neighbor is 4. Append 4 to the sequence, delete leaf 6.
6. Terminate algorithm since there are only two vertices left, 4 and 7. The resulting sequence is $(3, 3, 4, 4, 4)$.

**Example 1.5** Let us find a tree associated by the Prüfer's method to the sequence $(3, 3, 4, 4, 4)$.

1. The sequence has 5 elements, so $n = 5 + 2 = 7$.
2. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $S = (3, 3, 4, 4, 4)$
3. Initialize the graph as 7 isolated vertices.
4. The smallest element of $V$ not in $S$ is 1. The first element of $S$ is 3. Connect vertex 1 to vertex 3. Delete 1 from $V$ and 3 from $S$, so $V = \{2, 3, 4, 5, 6, 7\}$ and $S = (3, 4, 4, 4)$.
5. The smallest element of $V$ not in $S$ is 2. The first element of $S$ is 3. Connect vertex 2 to vertex 3. Delete 2 from $V$ and 3 from $S$, so $V = \{3, 4, 5, 6, 7\}$ and $S = (4, 4, 4)$.
6. The smallest element of $V$ not in $S$ is 3. The first element of $S$ is 4. Connect vertex 3 to vertex 4. Delete 3 from $V$ and 4 from $S$, so $V = \{4, 5, 6, 7\}$ and $S = (4, 4)$.
7. The smallest element of $V$ not in $S$ is 5. The first element of $S$ is 4. Connect vertex 5 to vertex 4. Delete 5 from $V$ and 4 from $S$, so $V = \{4, 6, 7\}$ and $S = (4)$.
8. The smallest element of $V$ not in $S$ is 6. The first element of $S$ is 4. Connect vertex 6 to vertex 4. Delete 6 from $V$ and 4 from $S$, so $V = \{4, 7\}$ and $S$ is empty.
9. $S$ is empty, so connect the two vertices in $V$, i.e. connect vertex 4 and vertex 7. The resulting tree is that of Figure 2 which is obviously the same as that of Figure 1.




$\longleftrightarrow (3, 3, 4, 4, 4)$

Figure 2: Example of the Prüfer's bijection between labeled trees on $n$ vertices and sequences of $n - 2$ elements from $\{1, 2, \ldots, n\}$ with repetitions allowed. In this example, $n = 7$.