Graph colorings

`Math 314`

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1 Vertex colorings

Definition 1.1 A *coloring* of a graph G is an assignment of colors to vertices of G so that no two adjacent vertices receive the same color.

Definition 1.2 The *chromatic number* of a graph G, denoted $\chi(G)$, is the least number of colors required for a coloring of G.

Note that the proof of the fact that $\chi(G) = k$ should consist of 2 parts:

- $\chi(G) \leq k$, i.e. there is a coloring of G with k colors.
- $\chi(G) \geq k$, i.e. any coloring of G requires at least k colors, i.e. $k-1$ colors is not enough to color G.

Example 1.3

$$\begin{gathered}
\chi(K_0) = 0, \\
\chi(\overline{K}_p) = 1 \text{ for } p \geq 1, \\
\chi(K_p) = p \text{ for } p \geq 1, \\
\chi(K_{m,n}) = 2 \text{ for } m \geq 1, n \geq 1, \\
\chi(C_{2n}) = 2, \quad \chi(C_{2n+1}) = 3, \\
\chi(P_n) = 2 \text{ for } n \geq 2, \\
\chi(T) = 2 \text{ for any tree } T \text{ with } \geq 2 \text{ vertices}, \\
\chi(G) \geq 2 \text{ for any graph } G \text{ with at least one edge.}
\end{gathered}$$

The following theorem gives an upper bound on the chromatic number of G.

Theorem 1.4 *For any graph G, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.*

Proof. There is an induction proof given in the textbook (see Section 9.2, Theorem 9.7, p. 207. We will give a different argument using a technique called the *Greedy Algorithm*.

First of all, without loss of generality, we may assume that G is connected (otherwise, we can color each connected component independently). Color the vertices of G with $\Delta(G)+1$ colors $1, 2, \ldots, \Delta(G)+1$, as follows.

1. Color any vertex of G with color 1.
2. While there is an uncolored vertex adjacent to a colored vertex, color such a vertex with the lowest color not yet assigned to any of its neighbors. (That's where we are being greedy).

We now need to prove that we cannot get stuck at any step of this algorithm. Any vertex v of G has $\leq \Delta(G)$ neighbors, so even if they are all colored with different colors, there are $\leq \Delta(G)$ colors used on the neighbors of v. But we have $\Delta(G)+1$ colors, so if v is uncolored, there will always be at least 1 color available for v.

Also since G is connected, there will always be uncolored vertex adjacent to a colored vertex after Step 1 until all vertices of G are colored.

This produces a coloring of G with $\Delta(G)+1$ colors, so $\chi(G) \leq \Delta(G)+1$. □

In fact, an even stronger result is true, but its proof is a bit more involved and will be omitted.

Theorem 1.5 (Brooks) *If G is neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

We can also start with $\chi(G)$ and determine some properties of G, or even G itself.

Example 1.6 Very low chromatic numbers:

1. $\chi(G) = 0$ implies that $G = \emptyset$, i.e. G is an empty graph (no vertices, no edges).
2. $\chi(G) = 1$ implies that $G = \overline{K}_p$ for some $p \geq 1$, i.e. G consists of isolated vertices only and has no edges.

Definition 1.7 A graph G is called *bipartite* if $\chi(G) \leq 2$.

Theorem 1.8 *A graph G is bipartite if and only if its set of vertices V can be partitioned into 2 subsets V_1 and V_2 (either one or both is allowed to be empty) so that each edge of G is between a vertex in V_1 and a vertex in V_2.*

Example 1.9 From the Example 1.3, we see that the empty graph, a graph with no edges, any path, any tree and any even cycle are bipartite.

The following theorem gives a criterion for a graph to be bipartite.

Theorem 1.10 *A graph G is bipartite if an only if G has no odd cycles.*

To prove a part of this theorem we will need the following definition.

Definition 1.11 The *distance* between two vertices x and y in a graph G is the length (number of edges) of the shortest path between x and y. If there is no such path, the distance is assumed to be infinite. The distance between x and y is denoted $d(x, y)$.

Distance between two points not connected by any path is assumed to be infinite. We note that this distance satisfies the same basic properties as the usual distance (in geometry), in other words:

1. $d(x, y) \geq 0$ for any x and y;

2. $d(x,x)=0$, and $d(x,y)=0$ implies $x=y$;
3. $d(x,y)=d(y,x)$ for any x and y;
4. $d(x,y)\leq d(x,z)+d(z,y)$ for any x, y and z (this is called the *triangle inequality*).

We will now continue with the proof of Theorem 1.10.

Proof. ($\Rightarrow$) Clearly, if a graph contains an odd cycle as a subgraph, then it cannot be bipartite, since coloring just the vertices of that odd cycle requires 3 colors. Therefore, if a graph is bipartite, then it contains no odd cycles.

($\Leftarrow$) Without loss of generality, we may assume that G is connected. Pick a vertex of G, and call it v_0. Now given any vertex x of G, color x RED if $d(v_0,x)$ is even, and color x BLUE if $d(v_0,x)$ is odd. We claim that this coloring is valid, in other words, that no two adjacent vertices of G will get the same color.

Why? Indeed, let us assume that our claim is false, i.e. there are two adjacent vertices x and y which got the same color. Let $P(x)$ be the shortest path from v_0 to x, and let $P(y)$ be the shortest path from v_0 to y. Then $P(x)$ has length $d(v_0,x)$ and $P(y)$ has length $d(v_0,y)$. Note that $P(y)\cup yx$ is also a path from v_0 to x, so it must be at least as long as $P(x)$. In other words, $d(v_0,x)\leq d(v_0,y)+1$. Similarly, $P(x)\cup xy$ is also a path from v_0 to y, so it must be at least as long as $P(y)$. In other words, $d(v_0,y)\leq d(v_0,x)+1$. Combining both inequalities, we get $-1\leq d(v_0,x)-d(v_0,y)\leq 1$, i.e. $|d(v_0,x)-d(v_0,y)|\leq 1$. But x and y have the same color, so $d(v_0,x)$ and $d(v_0,y)$ are both even or both odd, so $d(v_0,x)-d(v_0,y)$ is even. Therefore, $d(v_0,x)-d(v_0,y)=0$, i.e. $d(v_0,x)=d(v_0,y)$. Let w be the last common vertex of $P(x)$ and $P(y)$ starting from v_0. Then $d(v_0,x)=d(v_0,w)+d(w,x)$. Note that the triangle inequality becomes an equality here since $P(x)$ is the *shortest* path from v_0 to x, and w is on $P(x)$. Similarly, $d(v_0,y)=d(v_0,w)+d(w,y)$, so $d(w,y)=d(v_0,y)-d(v_0,w)=d(v_0,x)-d(v_0,w)=d(w,x)$. The circuit from w to x along $P(x)$, then to y along xy, then back to w along $P(y)$ has no repeated vertices (by choice of w), so it is a cycle. But this cycle has $d(w,x)+d(w,y)+1=2d(w,x)+1$ edges, i.e. it is an odd cycle. This contradicts our assumption that G has no odd cycles. Therefore, our claim is true, so our 2-coloring is indeed valid, and hence G is bipartite. □

1.1 Critical graphs

Definition 1.12 A graph G is *critical* if for all proper subgraphs H of G, we have $\chi(H)<\chi(G)$.

Example 1.13 Complete graphs K_n and odd cycles C_{2n+1} are critical graphs.

Theorem 1.14 *A critical graph is connected. Moreover, a nontrivial (not K_1) critical graph has no cut vertex.*

Proof. If G is disconnected or has a cut vertex, then we can find a proper subgraph of G with the same chromatic number as G (a connected component if G is disconnected, or a block if G has a cut vertex). □

Theorem 1.15 *A critical graph G has minimum degree $\delta(G)\geq\chi(G)-1$.*

Proof. Suppose that G has a vertex v with $\deg(v)<\chi(G)-1$. Since G is critical, the subgraph $G-u$ can be colored with $\chi(G)-1$ colors. The neighbors of v cannot have all $\chi(G)-1$ colors since v has less than $\chi(G)-1$ neighbors, so we can color v with one of the colors not used by the neighbors of v. Therefore, G can be colored with $\chi(G)-1$ colors, so $\chi(G)\leq\chi(G)-1$, which is impossible. Therefore, every vertex v of G has $\deg(v)\geq\chi(G)-1$, so $\delta(G)\geq\chi(G)-1$. □

Theorem 1.16 *Every graph G contains a critical subgraph G with $\chi(H) = \chi(G)$.*

Proof. Among subgraphs H of G with $\chi(H) = \chi(G)$, take one with the minimum number of edges and without isolated vertices (unless $G = \overline{K_n}$). □

Corollary 1.17 *If $\chi(G) = k$, then G has $\geq k$ vertices of degree $\geq k - 1$.*

Proof. Combine the Theorems 1.15 and 1.16. □

2 Edge-coloring

Definition 2.1 An *edge-coloring* of a graph is an assignment of colors to edges of the graph.

Definition 2.2 An *proper* edge-coloring of a graph is an edge-coloring such that no two adjacent edges receive the same color.

Note that in a proper edge-coloring, the edges that received each color (together with their endpoints) form a 1-regular subgraph (e.g., all-red, all-blue, etc.)

Definition 2.3 The *edge-chromatic number* of a graph G, denoted $\chi'(G)$, is the minimum number of colors required for a proper edge-coloring of G.

Example 2.4

$$\chi'(K_0) = 0, \chi(K_1) = 0, \chi'(K_2) = 1, \chi'(K_3) = 3, \chi'(K_4) = 3,$$
$$\chi'(C_{2n}) = 2, \quad \chi'(C_{2n+1}) = 3,$$
$$\chi'(P_n) = 2 \text{ for } n \geq 2,$$
$$\chi'(T) = 2 \text{ for any tree } T \text{ with } \geq 2 \text{ vertices},$$
$$\chi(G) \geq 1 \text{ for any graph } G \text{ with at least one edge}.$$

Theorem 2.5 *For any graph G, $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.*

Proof. The lower bound is an immediate consequence of the fact that some vertex of G has degree $\Delta(G)$, hence G has $\Delta(G)$ mutually adjacent edges, all of which should receive different colors in a proper edge-coloring. The upper bound is proved using Greedy Algorithm (coloring edges greedily) as in the Theorem 1.4. Each edge is adjacent to $\leq \Delta(G) - 1$ other edges at each of its endpoints. Thus, $1 + (\Delta(G) - 1) + (\Delta(G) - 1) = 2\Delta(G) - 1$ colors will always suffice for a proper edge-coloring of G. □

In fact, it is easy to see that this Greedy Algorithm requires way too many colors. As it turns out, a much stronger result is true.

Theorem 2.6 (Vizing) *For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

We will omit the proof of Vizing's theorem here.

Vizing's theorem implies that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. Therefore, to show that $\chi'(G) = \Delta(G)$, we only need to show that G has a proper $\Delta(G)$-edge-coloring. On the other hand, to show that $\chi'(G) = \Delta(G) + 1$, we need to prove that G has no $\Delta(G)$-edge-coloring.

An overwhelming majority of graphs has $\chi' = \Delta$ (as it happens, a random graph has $\chi' = \Delta$ with probability 1).

As we saw above, odd cycles are a family of graphs with $\chi' = \Delta + 1$, while even cycles have $\chi' = \Delta$. Petersen graph is 3-regular (so $\Delta = 3$), but not properly 3-edge-colorable (so $\chi' = 4$). We will find more edge-chromatic numbers below.

Theorem 2.7 *For any integer $n \geq 1$, $\chi'(K_{2n}) = 2n - 1$ (i.e. $\chi' = \Delta$).*

Proof. As we know, K_{2n} can be decomposed into $2n - 1$ 1-factors using a turning method. Note that no two edges in a 1-factor can be adjacent, so we may assign the same color to all edges of each 1-factor. This produces a $(2n - 1)$-edge-coloring of K_{2n} as desired. □

Theorem 2.8 *For any integer $n \geq 1$, $\chi'(K_{2n+1}) = 2n + 1$ (i.e. $\chi' = \Delta + 1$).*

Proof. Because K_{2n+1} has an odd number of vertices, it has no 1-factor. Any 1-regular subgraph of K_{2n+1} may contain $\leq n$ edges (and can't be spanning). If K_{2n+1} had a proper $2n$-edge-coloring, then it would have $\leq n(2n) = 2n^2$ edges. But K_{2n+1} has $(2n)(2n + 1)/2 = n(2n + 1) = 2n^2 + n > 2n^2$ edges. Therefore, K_{2n+1} does not have a proper $2n$-edge-coloring, so $\chi'(K_{2n+1}) = 2n + 1$. □

Theorem 2.9 *For any integers $m, n \geq 1$, $\chi'(K_{m,n}) = \max(m, n)$ (i.e. $\chi' = \Delta$).*

Proof. Without loss of generality, assume $m \leq n$. Label the vertices $u_0, \ldots, u_{m-1}, v_0, \ldots, v_{n-1}$, so $K_{m,n}$ has all possible edges u_iv_j. Then each $\deg(u_i) = n$ and $\deg(v_j) = m$ for any i, j. Color $K_{m,n}$ with colors $0, 1, \ldots, n-1$ as follows. Label each edge u_iv_j with color $(i + j) \mod n$ (i.e. the remainder from division of $i + j$ by n). It is not difficult to see that all edges adjacent to each u_i and to each v_j will get different colors in this way, so our n-edge-coloring is proper. □

Recall that Petersen graph is 3-regular with edge-chromatic number 4. It is also not Hamiltonian. We will see now that these properties are connected.

Theorem 2.10 *Let G be a 3-regular graph with edge-chromatic number 4. Then G is not Hamiltonian.*

Proof. Since G is 3-regular (so all vertices are odd), G must have an even number of vertices. Suppose G is Hamiltonian, then any Hamilton cycle of G is even, so we can color its edges properly with 2 colors, say red and blue. Now each vertex is incident with 1 red edge, 1 blue edge and 1 uncolored edge. Thus, the uncolored edges form a 1-factor of G, so we can color all of them with the same color, say green. Thus, G must be 3-edge-colorable, which is impossible. Therefore, G cannot be Hamiltonian. □

Finally, we will prove a theorem where a not necessarily proper edge-coloring is useful.

Theorem 2.11 *Let G be a 4-regular pseudograph. The G is decomposable into two 2-factors.*

Proof. If G has p vertices and q edges, then $2q = 4p$, i.e. $q = 2p$. All vertices of G are even, so G is Eulerian. Moreover, an Eulerian circuit of G has an even number of edges, $2p$. Thus, we can color the edges of an Eulerian circuit of G with two colors, say red and blue, so that the colors alternate along the circuit. Then each vertex of G will be adjacent to 2 red edges and 2 blue edges, or 1 red loop and 2 blue edges, or 2 red edges and 1 blue loop, or 1 red loop and 1 blue loop. Then the red subgraph of G and the blue subgraph G are both 2-regular and spanning, i.e. are 2-factors, so G is decomposable into two 2-factors. □

Edge-coloring (proper or not) can be used to prove many other results, e.g. in graph decomposition or Ramsey theory.