Math 304 **Midterm 2 (Take-home)** **Due 11/20/2003**

This exam is due Tuesday, October 7, in class. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

---

1. Use the method of characteristic roots to solve the following linear recurrences:

   (a) $a_n = 15a_{n-1} - 44a_{n-2}$ for $n \geq 2$; $a_0 = 2$, $a_1 = 1$.

   (b) $a_n = 12a_{n-2} + 16a_{n-3}$ for $n \geq 3$; $a_0 = 2$, $a_1 = 4$, $a_2 = 12$.

2. (a) Let $f(n)$ be the number of subsets of $[n] = \{1, 2, \ldots, n\}$ ($[0] = \emptyset$) that contain no two consecutive elements, for integer n. Find the recurrence that is satisfied by these numbers, then find $f(0)$ and $f(1)$ and conclude which famous sequence $f(n)$ really is. *Hint:* a subset with no two consecutive elements either (1) does not contain n or (2) contains n but does not contain $n - 1$.

   (b) *(extra credit)* For given integers n and k, determine the number $f(n, k)$ of k-subsets of $[n]$ that contain no two consecutive elements. *Hint:* given a k-subset $S = \{i_1 < i_2 < \cdots < i_k\}$ of $[n]$ there must be an element between any i_j and i_{j+1}, say $i_j + 1$, which is not in S. How many elements do you need to exclude?

   (c) *(extra credit)* By comparing the results of the previous two parts, deduce an identity.

3. Determine the number of permutations of length n which are simultaneously derangements and involutions. Try to find a closed formula for that number (i.e. have no dots "$\ldots$" in the answer).

4. Give a combinatorial proof that $\binom{2n}{n} < 4^n$. *Hint:* $4 = 2^2$.

5. Prove that the sequence of even Fibonacci numbers $\{F_0, F_2, F_4, F_6, \ldots, F_{2n}, \ldots\}$ satisfies the linear recurrence $F_{2n} = 3F_{2(n-1)} - F_{2(n-2)}$ for $n \geq 2$ with initial conditions $F_0 = 1$, $F_2 = 2$. *Hint:* Use the recurrence for the sequence of all Fibonacci numbers several times.