Math 304 **Midterm 1 (Take-home)** **Due 10/7/2003**

This exam is due Tuesday, October 7, in class. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. The set M consists of 9 positive integers each of which is a product of nonnegative powers of 2, 3 and 5. Prove that there are two elements in M whose product is a square of an integer. *Hint:* For each element $x = 2^i 3^j 5^k$ of M, consider parity of exponents i, j and k.

2. Prove that $7^n - 1 - (n+1)n(n-1)$ is divisible by 6 for any integer $n \geq 0$.

3. Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3. *Hint:* Use induction on the number of digits. It may be easier to prove a more general statement: a positive number and the sum of its digits are divisible by 3 with the same remainder.

4. A north-east integer lattice path in the plane is a sequence of unit steps $(1, 0)$ east and $(0, 1)$ north. Determine the number of north-east integer lattice paths between the origin $(0, 0)$ and the point (m, n) for nonnegative integers m and n. *Hint:* What is the number of steps north? steps east? all steps?

5. Find a nice, compact formula for the coefficient of $a^3 b^2 c d^4$ in the expansion of $(a + 5b + 2c + 2d)^{10}$, and compute this coefficient without doing the actual expansion.

6. You have 8 hours (a whole night!) to do homework for 4 classes. You know you must spend at least 1 hour on each subject. You decide to do homework one subject at a time and to allocate a whole (integer) number of hours to each subject. Determine the number of different such allocations.

7. (a) Let $g(n, k)$ be the number of k-compositions of n (we proved in class that $g(n, k) = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$, but you won't need that here). Give a combinatorial proof of the following formula:

   $$g(n, k) = g(n, k-1) + g(n-1, k).$$

   Hint: The last part of a composition is either 0 or at least 1.

   (b) We proved in class that $g(n, k) = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$. A simple substitution shows that $g(n, k) = g(k-1, n+1)$. Give a bijective proof that there are as many k-compositions of n as $(n+1)$-compositions of $k-1$. *Hint:* Dots and bars, bars and dots.