

1. (a) Prove that the Stirling number of the of first kind

$$c(n, n-2) = 3 \binom{n}{4} + 2 \binom{n}{3}.$$

- (b) Can you give a combinatorial proof of the above result?

2. Let  $f$  and  $g$  be two permutations of  $[n]$  (i.e. bijections from  $[n]$  to  $[n]$ ). Prove that the two compositions  $f \circ g : x \mapsto f(g(x))$  and  $g \circ f : x \mapsto g(f(x))$  have the same number of fixed points.

3. An *involution* is a permutation with cycles of lengths 1 or 2. In other words, a permutation  $\pi$  is an involution if and only if  $\pi^2 = \pi \circ \pi = id$ , the identity map. Let  $r(n)$  be the number of involutions of  $[n]$ . Prove that

$$r(n) = r(n-1) + (n-1)r(n-2).$$

4. (a) Use the sieve formula to find the number of permutations of  $[6]$  with descent set  $\{1, 3, 5\}$ , i.e. permutations  $\pi$  of  $[6]$  such that  $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \pi(5) > \pi(6)$ .

- (b) Use the sieve formula to find the number of permutations of  $[6]$  with descent set  $\{2, 4\}$ , i.e. permutations  $\pi$  of  $[6]$  such that  $\pi(1) < \pi(2) > \pi(3) < \pi(4) > \pi(5) < \pi(6)$ .

- (c) Give a bijective proof of why the answers in the previous two parts are equal.

5. Roberts, section 6.1, problems 12, 26, 29(a,d,e).

6. Roberts, section 6.2, problem 7.

*Hints:*

- 1a. Use the recurrence relation for  $c(n, k)$  and the fact that  $c(n, n-1) = \binom{n}{2}$ .

- 1b. Think of how  $n$  elements can be partitioned into  $n-2$  cycles. Use the fact that  $3 = c(4, 2)$  and  $2 = c(3, 1)$ .

2. Consider  $f(i)$  for each fixed point  $i$  of  $f \circ g$ .

3. The number  $n$  is either in a cycle by itself or in a cycle with some other element.

4. For each  $i = 1, 2, \dots, n-1$ , property  $i$  is “ $i$  is a descent,” i.e. “ $\pi(i) > \pi(i+1)$ .”

- 6.2.26 Without the property  $x_i \leq 7$ , which objects in the Twelffold Way is described here? Now use the sieve on these objects with property  $i$  being “ $x_i \geq 8$  ( $i = 1, 2, 3$ ).”

- 6.2.29 Let  $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$  for some distinct primes  $p_1, p_2, \dots, p_r$  and some positive integers  $a_1, a_2, \dots, a_r$ . To apply the principle of inclusion and exclusion, for each positive integer  $m \leq n$  let property  $i$  be  $p_i | m$  (i.e. “ $m$  is divisible by  $p_i$ ”). Determine how many of such integers  $m$  have at least all the properties in  $\{i_1, i_2, \dots, i_j\}$  and use the sieve formula.