Math 304 **Final Exam** **Due 12/17/2003**

This exam is due Wednesday, December 17, in my office, Carver 456, by 5pm. You may consult the texts for this course, the handouts, the homework solutions, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to PROVE YOUR ANSWERS and SHOW ALL WORK.

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1. Find the ordinary generating function for the *harmonic* numbers

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Hint: Recognize $\{H_n\}$ as the convolution of two simpler sequences. You may want to use the handout with the list of useful power series to find their generating functions.

2. Use the method of characteristic roots to solve the following recurrence relations:

   (a) $a_n = 3a_{n-1} - 2a_{n-2}$; $a_0 = 0$, $a_1 = 1$.

   (b) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$; $a_0 = 1$, $a_1 = 0$, $a_2 = 1$.

3. Find the coefficient of x^n in the power series of

$$f(x) = \frac{1}{(1-x^2)^2},$$

first by the method of partial fractions, and second, give a much simpler derivation by using a substitution.

4. Find an explicit formula for $\{g_n\}_{n=0}^{\infty}$ given by

$$g_n = -ng_{n-1} + \sum_k \binom{n}{k} g_k g_{n-k}, \quad n \geq 2; \qquad g_0 = 0,\ g_1 = 1.$$

Hint: You may want to use an exponential generating function and recall the exponential convolution. Alternatively, guess the exact formula for $f(n)$, then prove it by induction.

5. Find the formula for

$$\frac{d^n}{dx^n}\left(e^{e^x}\right),$$

the nth derivative of e^{e^x} (that's e raised to the power e^x). *Hint:* Differentiate e^{e^x} a few times, study the pattern, and conjecture the general form of the answer, including some constants to be determined. Then find a recurrence formula (using the fact that the $(n+1)$-st derivative is the derivative of the nth derivative) and the initial values for the constants in question, and identify them as some "famous" numbers that we have studied.

6. Prove that the two-variable generating function for $p(n,k)$, the number of partitions of n into k parts, is

$$P(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}p(n,k)x^n y^k=\prod_{m=1}^{\infty}\frac{1}{1-yx^m}$$

   Hint: Try a proof similar to the proof of

$$P(x)=\sum_{n=0}^{\infty}p(n)x^n=\prod_{m=1}^{\infty}\frac{1}{1-x^m}$$

   Note that the term yx^m corresponds to "1 part of size m."

7. The *ballot numbers* $b(n,k)$ are defined as

$$b(n,k)=\frac{k}{2n+k}\binom{2n+k}{n},\quad k\geq 1.$$

   (a) Show that the nth Catalan number $C_n=b(n,1)$, and

$$b(n,k)=\frac{k}{n+k}\binom{2n+k-1}{n},\quad b(n,k)=\binom{2n+k-1}{n}-\binom{2n+k-1}{n-1}.$$

   (b) *(extra credit)* Consider an election with 2 candidates, R and D, where R got n votes and D got $n+k-1$ votes, $k\geq 1$. After each vote was cast (for R or for D), we calculated by how many votes D was currently ahead of R, and it turned out that D was never ahead of R by k votes or more (i.e. was behind R, tied with R, or ahead of R by $\leq k-1$ votes). In other words, the running total of votes for D was always less than the running total of votes for R plus k. Prove that the number of possible sequences of votes which satisfy this condition is $b(n,k)$. *Hint:* Use the second formula in part (a) and a combinatorial proof similar to that of $C_n=\binom{2n}{n}-\binom{2n}{n-1}$.

   (c) *(extra extra credit)* Let $B_k(x)=\sum_{n=0}^{\infty}b(n,k)x^n$ and $C(x)=\sum_{n=0}^{\infty}C_nx^n$ be the ordinary generating functions for sequences $\{b(n,k)\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$, respectively. Prove that $B_k(x)=C(x)^k$.

8. *(extra credit)* A strictly increasing sequence $\{f(n)\}_{n=1}^{\infty}$ of positive integers (i.e. $0<f(1)<f(2)<f(3)<\dots$) satisfies the recurrence relation

$$f(f(n))=3n.$$

Find $f(304)$. *Hint:* Don't use generating functions. Start by finding $f(f(1))$ and $f(1)$, then use the recurrence relation and the fact that each term must be greater than the previous one to compute first few terms of the sequence (up to $f(6)$ at least, since the next key idea after $f(1)$ is $f(4)$ and $f(5)$). By the time you compute all terms up to $f(9)$ (actually, up to $f(18)$ may be even better), you should have the idea as to how the whole sequence behaves (or the general formula for $f(n)$). Now prove your conjecture by induction and find $f(304)$.