Math 307A **Sketch of Solutions to Midterm 2**

1. (a) If $2\vec{u}+3\vec{v}+4\vec{w}=5\vec{u}+6\vec{v}+7\vec{w}$, then vectors $\vec{u},\vec{v},\vec{w}$ must be linearly dependent.

   TRUE. Subtracting left side of the equation from its right side, we get $3\vec{u}+3\vec{v}+3\vec{w}=\vec{0}$. Since not all the coefficients on the left are 0 (in fact, none is), the vectors $\vec{u},\vec{v},\vec{w}$ are linearly dependent.

   (b) The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

   FALSE. It is a subset of $\mathbb{R}^3$ since a 3×4 matrix corresponds to a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

   (c) If the image of an $n\times n$ matrix A is all of $\mathbb{R}^n$, then A must be invertible.

   TRUE. See Summary 3.3.11 on page 130 of the textbook.

   (d) If A is a 5×6 matrix of rank 4, then the nullity of A is 1.

   FALSE. The nullity of A is $n-\text{rank}(A)=6-4=2$.

   (e) There are $n\times n$ matrices S and $A\neq I_n$ such that $S^{-1}AS=I_n$.

   FALSE. If $S^{-1}AS=I_n$, then multiplying both sides by S on the left and by S^{-1} on the right, we get $A=SI_nS^{-1}=SS^{-1}=I_n$.

   (f) If A is an $n\times n$ matrix such that $A^2=A$, $\vec{x}$ is a vector in $\mathbb{R}^n$, and $\vec{u}=\vec{x}-A\vec{x}$, then $\vec{u}\in\ker A$.

   TRUE. We have $A\vec{u}=A(\vec{x}-A\vec{x})=A\vec{x}-A^2\vec{x}=A\vec{x}-A\vec{x}=\vec{0}$, so $\vec{u}\in\ker A$.

   (g) If a subspace V of $\mathbb{R}^n$ contains none of the standard basis vectors $\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n$, then V consists of the zero vector only.

   FALSE. For example, the subspace $V=\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ in $\mathbb{R}^2$ (i.e. the line $x=y$) does not contain either $\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ or $\vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

   (h) If two vectors are linearly dependent, then one of them is a scalar multiple of the other.

   TRUE. Suppose vectors $\vec{u},\vec{v}$ are such that $a\vec{u}+b\vec{v}=\vec{0}$, and either $a\neq 0$ or $b\neq 0$. Then $\vec{u}=-\frac{b}{a}\vec{v}$ if $a\neq 0$, or $\vec{v}=-\frac{a}{b}\vec{u}$ if $b\neq 0$, so in either case one of the vectors is a scalar multiple of the other.

2. Given the matrix $A=\begin{bmatrix}1&2&3&2&1\\3&6&9&6&3\\1&2&4&1&2\\2&4&9&1&5\end{bmatrix}$, find rref$(A)$, a basis of im$(A)$, a basis of ker$(A)$, the rank of A, and the nullity of A.

   Solution. We start by finding rref(A).

$$A=\begin{bmatrix}1&2&3&2&1\\3&6&9&6&3\\1&2&4&1&2\\2&4&9&1&5\end{bmatrix}\to\begin{bmatrix}1&2&3&2&1\\0&0&0&0&0\\0&0&1&-1&1\\0&0&3&-3&3\end{bmatrix}\to\begin{bmatrix}1&2&3&2&1\\0&0&1&-1&1\\0&0&3&-3&3\\0&0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&2&0&5&-2\\0&0&1&-1&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}=\text{rref}(A)$$

We see that $\operatorname{rref}(A)$ has 2 leading 1's, so $\operatorname{rank}(A) = 2$. Also, A is a 4×5 matrix, so $\operatorname{nullity}(A) = 5 - \operatorname{rank}(A) = 5 - 2 = 3$.

The leading 1's of $\operatorname{rref}(A)$ are in the columns 1 and 3, so columns 1 and 3 of A, $\begin{bmatrix}1\\3\\1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\9\\4\\9\end{bmatrix}$, form a basis of $\operatorname{im}(A)$.

We will now find $\ker A$ and its basis. We have $\vec{x} \in \ker A \iff A\vec{x} = \vec{0} \iff \operatorname{rref}(A)\vec{x} = \vec{0}$. In other words,

$$\begin{bmatrix}1 & 2 & 0 & 5 & -2\\0 & 0 & 1 & -1 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \iff \begin{cases}x_1 + 2x_2 + 5x_4 - 2x_5 = 0\\x_3 - x_4 + x_5 = 0\end{cases} \iff \begin{cases}x_1 = -2x_2 - 5x_4 + 2x_5\\x_3 = x_4 - x_5\end{cases}$$

The free variables are x_2, x_4, x_5. Assigning to them arbitrary values $x_2 = r$, $x_4 = s$, $x_5 = t$, we get

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} \in \ker A \iff \vec{x} = \begin{bmatrix}-2r - 5s + 2t\\r\\s - t\\s\\t\end{bmatrix} = r\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-5\\0\\1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\-1\\0\\1\end{bmatrix}$$

Therefore,

$$\ker A = \operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-5\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\\1\end{bmatrix}\right\}$$

Since $\dim(\ker A) = \operatorname{nullity}(A) = 3$, these vectors form a minimal spanning set, i.e. a basis of $\ker A$.

3. Determine whether the polynomials $f_1(x) = x^2 + 4x + 3,\ f_2(x) = 3x^2 + 2x + 1,\ f_3(x) = 2x^2 + 3x + 2$ form a basis of the linear space P_2.

Solution. The coordinates of f_1, f_2, f_3 in the basis $\mathcal{B} = \{x^2, x, 1\}$ of P_2 are

$$[f_1]_{\mathcal{B}} = \begin{bmatrix}1\\4\\3\end{bmatrix}, \quad [f_2]_{\mathcal{B}} = \begin{bmatrix}3\\2\\1\end{bmatrix}, \quad [f_3]_{\mathcal{B}} = \begin{bmatrix}2\\3\\2\end{bmatrix}.$$

Since $\dim P_2 = 3$, and we have 3 vectors, it is enough to determine whether or not these vectors are linearly independent. To do that, we will see if $A = \begin{bmatrix}| & | & |\\ [f_1]_{\mathcal{B}} & [f_2]_{\mathcal{B}} & [f_3]_{\mathcal{B}}\\ | & | & |\end{bmatrix} = \begin{bmatrix}1 & 3 & 2\\4 & 2 & 3\\3 & 1 & 2\end{bmatrix}$ has rank 3, i.e. whether $\operatorname{rref}(A) = I_3$. Gauss-Jordan elimination yields

$$\begin{bmatrix}1 & 3 & 2\\4 & 2 & 3\\3 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 3 & 2\\0 & -10 & -5\\0 & -8 & -4\end{bmatrix} \to \begin{bmatrix}1 & 3 & 2\\0 & 1 & \frac{1}{2}\\0 & -8 & -4\end{bmatrix} \to \begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{1}{2}\\0 & 0 & 0\end{bmatrix} \neq I_3,$$

so the f_1, f_2, f_3 are linearly dependent, hence cannot form a basis of P_2.

4. Find the kernel of the linear transformation T from P_2 to $\mathbb{R}^2$ given by $T(f) = \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$ for any $f(t) = a + bt + ct^2$ in P_2.

Solution 1. Plugging in $t = 1$ and $t = 2$ into $f(t)$, we get $f(1) = a+b+c$, $f(2) = a+2b+4c$. Therefore,

$$f \in \ker T \iff T(f) = 0 \iff \begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} a + b + c = 0 \\ a + 2b + 4c = 0 \end{cases} \iff \begin{cases} a = 2c \\ b = -3c \end{cases}$$

i.e. $f(t) = 2c - 3ct + ct^2 = c(2 - 3t + t^2)$, where c is an arbitrary constant. Thus,

$$\ker T = \{c(2 - 3t + t^2) \mid c = \text{any constant}\}.$$

Solution 2. We have

$$f \in \ker T \iff T(f) = 0 \iff \begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} f(1) = 0 \\ f(2) = 0 \end{cases}$$

i.e. if both $t = 1$ and $t = 2$ are zeros of $f(t)$. Since $f(t)$ is a quadratic polynomial with zeros 1 and 2, it must have the form $f(t) = c(t-1)(t-2) = c(t^2 - 3t + 2)$, where c is any constant. Conversely, for any such polynomial $f(t)$, we have $f(1) = 0$ and $f(2) = 0$, so $T(f) = 0$, i.e. $f \in \ker T$. Therefore,

$$\ker T = \{c(2 - 3t + t^2) \mid c = \text{any constant}\}.$$

5. Find the basis of the space of all 2×2 matrices $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Solution. Computing the products on the left and on the right, we obtain

$$\begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

In other words,

$$\begin{cases} a = d \\ -b = c \\ -c = b \\ d = a \end{cases} \iff \begin{cases} d = a \\ c = -b \end{cases},$$

so $S = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, where a and b are arbitrary. Thus, a basis of our space consists of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

6. A linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by $T(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \vec{x}$ in the standard basis $\vec{e}_1, \vec{e}_2$ of $\mathbb{R}^2$. What is the matrix of T in the new basis $\vec{f_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{f_2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$?

Solution. The coordinates of the new basis vectors f_1, f_2 in the old basis e_1, e_2 are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, so the matrix of these coordinates is $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Therefore, the matrix of T in the new basis f_1, f_2 is

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = I_2 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$