**Math 307** **Sketch of Solutions to Midterm 1** **10/18/2002**

1. (a) A system with fewer unknowns than equations must have infinitely many solutions or none.
FALSE. Such a system may have a unique solution with some redundant equations (in this case, the rref of the coefficient matrix of such a linear system is $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$, i.e. an identity matrix on top of a zero matrix).

(b) If $A$ and $B$ are any two $n \times n$ matrices of rank $n$, then $A$ can be transformed into $B$ by means of elementary row transformations.
TRUE. Both $A$ and $B$ may be transformed into the identity matrix $I_n$ by a sequence of elementary row transformations. Elementary row transformations are invertible linear transformations, so we can "undo" them in reverse order to transform $I_n$ into $A$ or $B$ as well. Thus, we may transform $A$ into $B$ by transforming $A$ into $I_n$, then $I_n$ into $B$.

(c) The system $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent for all $4 \times 3$ matrices $A$.

FALSE. For example, let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) There is a nonzero upper triangular $2 \times 2$ matrix $A$ such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
TRUE. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{then} \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(e) The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x+3)^2 - (x-3)^2 \end{bmatrix}$ is a linear transformation.

TRUE. Simplifying, we get $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x+3)^2 - (x-3)^2 \end{bmatrix} = \begin{bmatrix} 2y \\ 6x \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so $T$ is linear.

(f) If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible.

FALSE. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible, but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not.

(g) Matrix $A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a shear.

TRUE. Consider $A\vec{x} - \vec{x} = A\vec{x} - I\vec{x} = (A - I)\vec{x}$. We have $A - I = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$, so $(A - I) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 2x_2 \\ -2x_1 + 2x_2 \end{bmatrix} = (-2x_1 + 2x_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is parallel to the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $\vec{x}$ is parallel to $\vec{v}$, then $\vec{x}$ is a multiple of $\vec{v}$, so $x_1 = x_2$ and hence $A\vec{x} - \vec{x} = (A - I)\vec{x} = \vec{0}$, so $A\vec{x} = \vec{x}$. Thus, $A$ does represent a shear parallel to $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(h) For any two $n \times n$ matrices $A$ and $B$, we have $(A + B)^2 = A^2 + 2AB + B^2$.
FALSE. $(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$ when $AB \neq BA$ (which is usually the case).

SHOW ALL WORK in the remaining problems. Partial credit will be given. Each problem is worth 10 points.

2. Find all solutions of the following linear system: $\begin{cases} x_2 + 2x_4 + 3x_5 = 0 \\ \qquad 4x_4 + 8x_5 = 0 \end{cases}$

Perform the Gauss-Jordan elimination.

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 4 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & 1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & 1 & 2 & | & 0 \end{bmatrix}$$

Hence,

$$\begin{cases} x_2 - x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \iff \begin{cases} x_2 = x_5 \\ x_4 = -2x_5 \end{cases}$$

The free variables are $x_1, x_3, x_5$, so the set of solutions consists of all

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r \\ t \\ s \\ -2t \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad r, s, t = \text{any real numbers.}$$

3. Find $AB - BA$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

$$AB - BA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

4. Find all $2 \times 2$ diagonal matrices $A$ that satisfy $A^2 - A - 2I_2 = 0$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then $A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$, so we have

$$0 = A^2 - A - 2I_2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a^2 - a - 2 & 0 \\ 0 & b^2 - b - 2 \end{bmatrix}.$$

Therefore, $a^2 - a - 2 = 0$ and $b^2 - b - 2 = 0$, so $a = 2, -1$ and $b = 2, -1$. Thus, we have 4 solutions:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5. Use the Gauss-Jordan method to find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

$$[A|I_3] = \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 2 & | & 1 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & | & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = [I_3|A^{-1}],$$

so

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

6. Find the matrix $A$ of the linear transformation $T$ with

$$T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$

We have $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$ and $A\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$. Since the left factor $A$ is the same, we can combine these two equations into one:

$$A\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}.$$

Multiply both sides on the right by $\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}^{-1}$ to obtain

$$A = AI_2 = A\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}^{-1} = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}^{-1} = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}\frac{1}{1\cdot 5 - 2\cdot 2}\begin{bmatrix}5 & -2\\-2 & 1\end{bmatrix} = \begin{bmatrix}8 & -3\\-1 & 1\end{bmatrix}.$$

7. Find the matrix $A \neq \pm I_2$ of a rotation $T$ in the plane such that $A^4 = I_2$. (*Hint:* $I_2$ is also the matrix of rotation through angle $\pm 2\pi$. There are 2 possible answers.)

   Let $A$ be a matrix of rotation through angle $\alpha$. Thus, multiplying $A$ by any vector in $\mathbb{R}^2$ rotates this vector by angle $\alpha$. Since $A^4\vec{x} = A(A(A(A\vec{x})))$, it follows that multiplying $A^4$ by any vector rotates this vector 4 times by angle $\alpha$, which is the same as rotation by $4\alpha$. On the other hand, $I_2$ is a rotation by an angle $\pm 2\pi$ (as well as by angle 0 or any integer multiple of $2\pi$). Since $A^4 = I_2$, we have $4\alpha = \pm 2\pi$, i.e. $\alpha = \pm\frac{\pi}{2}$. Recall that $\cos(\pm\frac{\pi}{2}) = 0$ and $\sin(\pm\frac{\pi}{2}) = \pm 1$. Therefore,

$$A = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \text{ or } \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}.$$