1. Let $c_n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k}$. The sum on the right is exactly the convolution of the sequence $\{\binom{2n}{n}\}$ with itself, i.e. $\{c_n\} = \{\binom{2n}{n}\} * \{\binom{2n}{n}\}$, so

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} \binom{2n}{n} x^n\right)\left(\sum_{n=0}^{\infty} \binom{2n}{n} x^n\right) = \frac{1}{\sqrt{1-4x}}\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{n=0}^{\infty} 4^n x^n,$$

so $c_n = 4^n$.

2. Let $e^{at} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, $e^{bt} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$, $e^{(a+b)t} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$, then $e^{(a+b)t} = e^{at}e^{bt}$ implies that

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k},$$

i.e. $\{c_n\}$ is the exponential convolution of $\{a_n\}$ and $\{b_n\}$. But

$$e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = \sum_{n=0}^{\infty} a^n \frac{t^n}{n!},$$

so $a_n = a^n$. Similarly, $b_n = b^n$ and $c_n = (a+b)^n$, so we obtain

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

which is the binomial expansion of $(a+b)^n$.

3. Let S be the set of all north-east integer lattice paths from $(0,0)$ to $(5,5)$. Let A be the set of paths in S which pass through point $(1,1)$, and let B be the set of paths in S which pass through point $(4,4)$. We want to find the number of paths in S which are not in $A \cup B$ (not those that pass through either $(1,1)$ or $(4,4)$. So, we want to find $|S| - |A \cup B|$ ($|S|$ denotes the number of elements of the set S). But $|A \cup B| = |A| + |B| - |A \cap B|$, where the set $A \cap B$ consists of paths in S which pass through both $(1,1)$ and $(4,4)$. Thus, we we want to find

$$|S| - |A \cup B| = |S| - (|A| + |B| - |A \cap B|) = |S| - |A| - |B| + |A \cap B|.$$

The number of paths in S is the number of north-east integer lattice paths from $(0,0)$ to $(5,5)$, i.e.

$$|S| = \binom{5+5}{5} = \binom{10}{5} = 252.$$

Each path in A consists of a path from $(0,0)$ to $(1,1)$ followed by a path from $(1,1)$ to $(5,5)$, so

$$|A| = \binom{1+1}{1}\binom{(5-1)+(5-1)}{5-1} = \binom{2}{1}\binom{8}{4} = 2 \cdot 70 = 140.$$

Similarly, each path in B consists of a path from $(0,0)$ to $(4,4)$ followed by a path from $(4,4)$ to $(5,5)$, so

$$|B| = \binom{4+4}{4}\binom{(5-4)+(5-4)}{5-4} = \binom{8}{4}\binom{2}{1} = 70 \cdot 2 = 140.$$

Finally, each path in $A \cap B$ consists of a path from $(0,0)$ to $(1,1)$ followed by a path from $(1,1)$ to $(4,4)$, followed by a path from $(4,4)$ to $(5,5)$, so

$$|A| = \binom{1+1}{1}\binom{(4-1)+(4-1)}{4-1}\binom{(5-4)+(5-4)}{5-4} = \binom{2}{1}\binom{6}{3}\binom{2}{1} = 2 \cdot 20 \cdot 2 = 80.$$

Thus, the number of paths from $(0,0)$ to $(5,5)$ that avoid $(1,1)$ and $(4,4)$ is

$$|S| - |A| - |B| + |A \cap B| = 252 - 140 - 140 + 80 = 52.$$

4. Since the frog starts and ends at 0, the number of leaps forward and backward it took must be the same, namely, half the total number of leaps, i.e. $2n/2 = n$.

   Denote a leap forward by F and a leap backward by B. If the frog never lands on a negative number then the sequence of its leaps is a sequence of n F's and n B's such that, moving left to right along the sequence, the number of B's is never greater than the number of F's. Change B to N and F to E to get exactly the sequences of north-east integer lattice paths from $(0,0)$ to (n,n) that stay on or below the diagonal $x = y$. The number of such paths, as we know, is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

   Now the number of possible leap sequences of n F's and n B's is the same as the total number of paths from $(0,0)$ to (n,n), i.e. $\binom{n+n}{n} = \binom{2n}{n}$.

   Thus, the probability that the frog never lands on a negative number is

$$\frac{C_n}{\binom{2n}{n}} = \frac{1}{n+1}.$$

5. (a) There are $\binom{n}{k}$ ways to pick the first k integers (to be arranged in decreasing order. Now there are no restrictions on the last $n-k$ elements of the sequence, so we can permute them in $(n-k)!$ ways. Thus, the number of permutations of $[n]$ in which the first k elements are in decreasing order is

$$\binom{n}{k}(n-k)! = \frac{n!}{k!}.$$

   (b) To find the probability that the first k elements of a random permutation $[n]$ are in decreasing order, we must divide the answer from part (a) by the total number of permutations of $[n]$, i.e. by $n!$, to get

$$Prob(\text{first } k \text{ elements are in decreasing order}) = \frac{n!/k!}{n!} = \frac{1}{k!}.$$

   Now we want to find the number of permutations of $[n]$ whose initial decreasing subsequence has length exactly k. Those are exactly the permutations where the first k elements of a random permutation $[n]$ are in decreasing order, but the first $k+1$ elements are not. Thus, the number of these permutations is

$$\frac{n!}{k!} - \frac{n!}{(k+1)!}$$

   for $k < n$, and $n!/n! = 1$ for $k = n$, so the probability that the initial decreasing subsequence of a random permutation of $[n]$ has length exactly k is

$$\frac{\frac{n!}{k!} - \frac{n!}{(k+1)!}}{n!} = \frac{1}{k!} - \frac{1}{(k+1)!}$$

   for $k < n$, and $1/n!$ for $k = n$.

   (c) Given the answers to (b), the average size of the initial decreasing subsequence of a permutation in $[n]$ is

$$\begin{aligned} s_n &= \frac{\sum_{k=1}^{n-1} k\left(\frac{n!}{k!} - \frac{n!}{(k+1)!}\right) + n \cdot \frac{n!}{n!}}{n!} = \sum_{k=1}^{n-1} k\left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) + n\frac{1}{n!} \\ &= \frac{1}{1!} - \frac{1}{2!} + 2\frac{1}{2!} - 2\frac{1}{3!} + 3\frac{1}{3!} - 3\frac{1}{4!} + \cdots + (n-1)\frac{1}{(n-1)!} - (n-1)\frac{1}{n!} + n\frac{1}{n!} \\ &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!} \end{aligned}$$

Hence,

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e.$$

6. Let $f(n)$ be the number of subsets of $[n]$ with no two consecutive elements. Separate this subsets into two types: those that don't contain the element n and those that do. The subsets of the first type are exactly the subsets of $[n-1]$ with no two consecutive elements, so their number is $f(n-1)$. Each subset of the second type contains n, so cannot contain $n-1$. Therefore, it consists of n and any subset of $[n-2]$ with no two consecutive elements. Therefore, the number of the subsets of the second type is $f(n-2)$. Thus, we have

$$f(n) = f(n-1) + f(n-2), \quad n \geq 2.$$

The initial values of $\{f(n)\}$ are $f(0) = 1$ (the counted subset is $\emptyset$, the empty set) and $f(1) = 2$ (the counted subsets are $\emptyset$ and $\{1\}$). Note that $f(0) = 1 = F_1$ and $f(1) = 2 = F_2$, where F_n is the nth Fibonacci number. Also note that the Fibonacci numbers F_{n+1} satisfy the same recurrence relation as $f(n)$ and have the same initial values. Therefore, $f(n) = F_{n+1}$ for all $n \geq 0$.