Math 304 **Midterm 2 (Take-home)** **Due 12/5/2002**

This exam is due Thursday, December 5, in class. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. (a) Use generating functions to prove the following identity:

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

   (b) (*Extra credit*) Try to find a combinatorial proof of the above formula. *Warning:* This is *really* hard.

2. Derive the binomial expansion (Theorem 2.7) by looking at the coefficients of $t^n/n!$ in the power series expansion of each side of the identity

$$e^{(a+b)t} = e^{at}e^{bt}.$$

   Hint: Exponential convolution.

3. Determine the number of north-east integer lattice paths from $(0,0)$ to $(5,5)$ that avoid the points $(1,1)$ and $(4,4)$.

4. A frog takes leaps along the real line, each leap being of length 1 either to the left or to the right. Suppose that the frog starts at 0 and ends up back at 0 after $2n$ steps. What is the probability that the frog never landed on a negative number during its trip?

5. (a) Determine the number of permutations of $[n]$ in which the first k elements are in decreasing order. *Hint:* Note that nothing is assumed about the rest of the sequence (so the initial decreasing subsequence may or may not continue beyond the first k integers).

   (b) Find the probability that the first k elements of a random permutation $[n]$ are in decreasing order. Find the probability that the initial decreasing subsequence of a random permutation of $[n]$ has length exactly k (in other words, the first k elements are in decreasing order, but the $(k+1)$-st element is an increase over the kth element).

(c) (*Extra credit*) If we have a collection of objects of various nonnegative integer sizes, and for each integer $n \geq 0$, there are a_n objects of size n, then the *average* size of an object in our collection is defined by

$$\frac{\sum_n n a_n}{\sum_n a_n}.$$

(The denominator is simply the total number of objects in our collection.) Find the average size s_n of the initial decreasing subsequence of a permutation of $[n]$ (make sure you simplify the formula for s_n as much as possible). Find $\lim_{n\to\infty} s_n$.

6. (*Extra credit*) Let $f(n)$ be the number of subsets of $[n]$ which no two consecutive elements (in other words, such a subset cannot contain both 1 and 2, both 2 and 3, both 3 and 4, etc., both $n-1$ and n). Find the recurrence relation along with the initial conditions satisfied by $f(n)$. Use your result to identify $f(n)$ as some "famous" numbers we studied. *Hint:* A subset of $[n]$ with no two consecutive elements either (1) does not contain n, or (2) contains n and does not contain $n-1$. Express the number of subsets in each of these two piles in terms of $f(n)$.