

1. Write each element  $x$  in the set  $M$  as a product  $x = 2^i 3^j 5^k$  for some nonnegative integers  $i, j, k$  (which depend on  $x$ ). Consider the list of parities of  $i, j, k$  (i.e. whether each is odd or even), in that order. Put elements  $x_1 = 2^{i_1} 3^{j_1} 5^{k_1}$  and  $x_2 = 2^{i_2} 3^{j_2} 5^{k_2}$  in the same pigeonhole if the lists of parities of  $(i_1, j_1, k_1)$  and  $(i_2, j_2, k_2)$  are the same. There are 2 choices for the parity of each of the 3 components, for the total of  $2^3 = 8$  possible pigeonholes. Since  $M$  contains 9 elements, at least 2 of its elements will have the same list of parities. If  $x_1$  and  $x_2$  are these 2 elements, consider  $x_1 x_2 = 2^{i_1} 3^{j_1} 5^{k_1} 2^{i_2} 3^{j_2} 5^{k_2} = 2^{i_1+i_2} 3^{j_1+j_2} 5^{k_1+k_2}$ . If two integers have the same parity (i.e. are both odd or both even), then their sum is an even integer, hence  $i_1 + i_2 = 2\ell, j_1 + j_2 = 2m, k_1 + k_2 = 2n$  for some nonnegative integers  $\ell, m, n$ . Therefore,  $x_1 x_2 = 2^{2\ell} 3^{2m} 5^{2n} = (2^\ell 3^m 5^n)^2$ , which is a square of an integer.
2. (a) We will prove by induction on  $r$  that  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$  for all  $r \geq 0$  and  $n \geq 0$ .

*Induction Base.* Let  $r = 0$ . Then  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n}{0} = 1 = \binom{n+0+1}{0} = \binom{n+r+1}{r}$ , so our statement is true for  $r = 0$  and any  $n \geq 0$ .

*Induction Step.* Now let  $r$  be any nonnegative integer and suppose that our statement is true for  $r$ , i.e.  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$  for any  $n \geq 0$ . We must prove that our statement is true for  $r + 1$ , i.e.  $\sum_{k=0}^{r+1} \binom{n+k}{k} = \binom{n+r+2}{r+1}$  (in other words, we must prove that we can substitute  $r + 1$  for  $r$ , and the identity will still be true). We have

$$\begin{aligned} \sum_{k=0}^{r+1} \binom{n+k}{k} &= \sum_{k=0}^r \binom{n+k}{k} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} && \text{by the induction hypothesis} \\ &= \binom{n+r+2}{r+1} && \text{by the binomial formula} \end{aligned}$$

Therefore, our statement is true for  $r + 1$  as well.

Thus, by induction on  $r$ ,  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$  for any integers  $r, n \geq 0$ .

- (b) The combinatorial proof of  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$  is as follows. Consider an  $(n + r + 1)$ -bit sequence of zeros and ones which has exactly  $r$  zeros. There are  $\binom{n+r+1}{r}$  ways to choose places for the ones, hence, there are  $\binom{n+r+1}{r}$  such sequences. Such a sequence has  $(n + r + 1) - r = n + 1$  ones, hence at least 1 one (as  $n + 1 \geq 1$ ). Partition the set of these sequences according to the position of the rightmost one. The rightmost,  $(n + 1)$ -st, one cannot occupy any position from 1 to  $n$ , hence must occupy one of the positions from  $n + 1$  to  $n + r + 1$ . Say it occupies position  $n + k + 1$  for some  $k$ , where  $0 \leq k \leq r$ . Then there are only zeros to its right (so this part of the sequence is uniquely determined), and to its left there are  $n + k$  bits exactly  $n$  of which are ones. Thus, there are  $\binom{n+k}{k}$  ways to choose such a sequence. Therefore, there are  $\binom{n+k}{k}$  of  $(n + r + 1)$ -bit sequences of zeros and ones with exactly  $r$  zeros and the rightmost one at position  $n + k + 1$ . Since  $k$  must be one of the integers in  $\{0, 1, \dots, r\}$ , the total number of  $(n + r + 1)$ -bit sequences of zeros and ones with exactly  $r$  zeros is  $\sum_{k=0}^r \binom{n+k}{k}$ . But we proved above that the total is  $\binom{n+r+1}{r}$ . Therefore,  $\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$  for any  $n, r \geq 0$ .
3. To show that the sequence  $\left\{ \binom{n}{k} \right\}_{k=0}^n$  is unimodal, consider the ratios  $\frac{\binom{n}{k+1}}{\binom{n}{k}}$  for  $k = 0, 1, \dots, n - 1$ . We have

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{k!}{(k+1)!} \frac{(n-k)!}{(n-k-1)!} = \frac{1}{k+1} (n-k) = \frac{n-k}{k+1}.$$

Now

$$\frac{n-k}{k+1} \geq 1 \iff n-k \geq k+1 \iff 2k \leq n-1 \iff k \leq \frac{n-1}{2},$$

so  $\frac{\binom{n}{k+1}}{\binom{n}{k}} \geq 1 \iff k \leq \frac{n-1}{2}$ , i.e.  $\binom{n}{k+1} \geq \binom{n}{k} \iff k \leq \frac{n-1}{2}$ . Similarly,  $\binom{n}{k+1} < \binom{n}{k} \iff k > \frac{n-1}{2}$ , and the smallest integer  $k$  for which  $\binom{n}{k+1} < \binom{n}{k}$  is the smallest integer greater than  $\frac{n-1}{2}$ , i.e.  $\lceil \frac{n-1}{2} \rceil$ . It is easy to check that  $\lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$  (consider two cases: even  $n$  and odd  $n$ ). Therefore, our sequence is unimodal and its largest term is  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ . It is also easy to check (again, separately for even  $n$  and for odd  $n$ ) that  $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ , so  $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$ .

4. The coefficient of  $a^3b^2cd^4$  in the multinomial expansion of  $(a+b+c+d)^{10}$  is the multinomial coefficient

$$\binom{10}{3, 2, 1, 4} = \frac{10!}{3!2!1!4!} = 12600.$$

5. This is an exercise in name recognition. We want to count the number of ways to write 10 as an ordered sum of 5 positive integers, the summands being the numbers of spells used at levels 1, 2, 3, 4, 5. Hence, spells are indistinguishable, while levels are distinguishable, and since each summand is at least 1, we are counting surjective functions from spells to levels. Thus, the total number of such ordered sums is  $\binom{10-1}{5-1} = \binom{9}{4} = 126$ .