Math 304 **Midterm 1 (Take-home)** **Due 10/8/2002**

This exam is due Tuesday, October 8, in class. You may consult the text for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. The set M consists of 9 positive integers each of which is a product of nonnegative powers of 2, 3 and 5. Prove that there are two elements in M whose product is a square of an integer. *Hint:* For each element $x = 2^i 3^j 5^k$ of M, consider parity of exponents i, j and k.

2. (a) Prove that
$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+r}{r} = \binom{n+r+1}{r}.$$
Hint: Fix n and do an induction proof on r. Recall the binomial formula (Theorem 2.2).

   (b) *(Extra Credit)* Give a combinatorial proof of the above formula.

3. Note that the entries in any row of the Pascal's triangle (i.e. the sequence of numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ for a fixed n) increase for a while and then decrease. A sequence of numbers is called *unimodal* if there is an integer t (not necessarily unique) such that $a_0 \leq a_1 \leq \cdots \leq a_t \geq a_{t+1} \geq \cdots \geq a_n$.

   (a) Show that the sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ (i.e. $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$) is unimodal. *Hint:* Consider the ratios $\frac{\binom{n}{k+1}}{\binom{n}{k}}$.

   (b) Show that the largest entry in the above sequence is $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$. *Hint:* You may want to consider the cases of even n and odd n.

4. Find a nice, compact formula for the coefficient of $a^3b^2cd^4$ in the expansion of $(a+b+c+d)^{10}$, and compute this coefficient without doing the actual expansion.

5. The game of "Gloom" has 5 levels. In each level, you must get past one gloomy ogre. Your only weapon is the happy spell machine which shoots happy spells at gloomy ogres. You need at least 1 happy spell (i.e. 1 or more) to get past each gloomy ogre. In how many ways can you allocate your supply of 10 happy spells by level so as to use all of them by the time you get through level 5?