1. *Solution 1.* We have

$$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4)^n x^n,$$

so we need to prove that

$$\binom{-\frac{1}{2}}{n}(-4)^n = \binom{2n}{n}.$$

But

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} \\ &= \left(-\frac{1}{2}\right)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!} = \left(-\frac{1}{2}\right)^n \frac{(2n)!}{(2\cdot 4\cdot 6\cdots(2n))\,n!} = \left(-\frac{1}{2}\right)^n \frac{(2n)!}{2^n n! n!} \\ &= \left(-\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n \frac{(2n)!}{n!(2n-n)!} = \left(-\frac{1}{4}\right)^n \binom{2n}{n},\end{aligned}$$

so we are done.

Solution 2. We know that

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^n$$

Multiplying both sides by x, we get

$$\frac{1-\sqrt{1-4x}}{2} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^{n+1}.$$

Now taking derivative of both sides, we obtain on the left

$$\frac{d}{dx}\left(\frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}}\right) = 0 - \frac{1}{2}\cdot\frac{1}{2}(1-4x)^{\frac{1}{2}-1}(-4) = (1-4x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-4x}}$$

and on the right

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^{n+1}\right) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{d}{dx}\left(\frac{1}{n+1} x^{n+1}\right) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n,$$

so

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n,$$

as desired.

2. (a) We have

$$\begin{aligned} b(n,k) &= \frac{k}{2n+k}\binom{2n+k}{n} = \frac{k}{2n+k}\frac{(2n+k)!}{(n+k)!n!} = k\frac{(2n+k-1)!}{(n+k)!n!} \\ &= \frac{k}{n+k}\frac{(2n+k-1)!}{(n+k-1)!n!} = \frac{k}{n+k}\binom{2n+k-1}{n},\end{aligned}$$

so

$$b(n,1) = \frac{1}{n+1}\binom{2n+1-1}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

Furthermore,

$$\begin{aligned}\binom{2n+k-1}{n}-\binom{2n+k-1}{n-1} &= \frac{(2n+k-1)!}{(n+k-1)!n!}-\frac{(2n+k-1)!}{(n+k)!(n-1)!}\\ &= \frac{(2n+k-1)!}{(n+k)!n!}((n+k)-n)=\frac{k(2n+k-1)!}{(n+k)!n!}\\ &= \frac{k}{2n+k}\frac{(2n+k)!}{(n+k)!n!}=\frac{k}{2n+k}\binom{2n+k}{n}=b(n,k).\end{aligned}$$

(b) Let $r(i)$ and $d(i)$ be the number of votes for R and D, respectively, after the first i votes were cast. Then $r(i) \geq 0$, $d(i) \geq 0$, $r(i)+d(i)=i$ for all $i=0,1,2,\dots,2n+k-1$, $r(0)=d(0)=0$, $r(2n+k-1)=n+k-1$, $d(2n+k-1)=n$, and $r(i)-d(i) \leq k-1$ for all $i \leq 2n+k-1$. Furthermore, after each vote, either $r(i)$ increases by 1 and $d(i)$ stays the same, or $r(i)$ stays the same and $d(i)$ increases by 1.

The total number of possible vote sequences of $n+k-1$ votes for R and n votes for D is $\binom{n+(n+k-1)}{n}=\binom{2n+k-1}{n}$, since choosing n D-votes out of the total of $2n+k-1$ votes means that the rest are R-votes.

Therefore, we only need to show that there are $\binom{2n+k-1}{n-1}$ vote sequences where $r(i)-d(i) \geq k$ for *some* $i \leq 2n+k-1$. Then the number of vote sequences where $r(i)-d(i) \leq k-1$ for all $i \leq 2n+k-1$ is $\binom{2n+k-1}{n}-\binom{2n+k-1}{n-1}=b(n,k)$ from part (a).

Consider a "bad" vote sequence, i.e. one where R does lead D by at least k votes at some point. Suppose the *first* time this happens is after the ith vote was cast. Then $r(i)-d(i)=k$ (the least integer greater than $k-1$) and $r(j)-d(j) \leq k-1$ for all $j<i$.

Let $r(i)=v$, then $d(i)=v-k$, so out of the first i votes, R got v votes and D got $v-k$ votes. Therefore, R got $(n+k-1)-v$ of the remaining votes, and D got $n-(v-k)=n+k-v$ of the remaining votes.

Now (this is the crucial trick in the proof) exchange the votes for R and D after the ith vote. In the new vote sequence, R gets the total of $v+(n+k-v)=n+k$ votes, while D gets $(v-k)+(n+k-1-v)=n-1$ votes. Clearly, the number of vote sequences where R gets $n+k$ votes and D gets $n-1$ votes is $\binom{n+k+n-1}{n-1}=\binom{2n+k-1}{n-1}$.

Conversely, any such new sequence can be converted to a "bad" old sequence of votes with $n+k-1$ R-votes and n D-votes. Why? Well, in the new sequence R leads D by 0 votes at the beginning and by $(n+k)-(n-1)=k+1$ votes at the end. So R must lead D by k votes at some point. Let ith vote be the *first* such point in the voting. Exchange votes for R and D as above to get a "bad" old sequence. The argument is the same as in the preceding paragraph. Thus, the number of "bad" old vote sequences is the same as the number of the new vote sequences, i.e. $\binom{2n+k-1}{n-1}$. This ends the proof.

3. This trick can be used for any sequence, not just Fibonacci numbers. Let $\{f_n\}_{n=0}^{\infty}$ be any sequence, and let $f(x)=\sum_{n=0}^{\infty} f_n x^n$. Then $f(-x)=\sum_{n=0}^{\infty} f_n(-x)^n=\sum_{n=0}^{\infty}(-1)^n f_n x^n$, so

$$f(x)+f(-x)=\sum_{n=0}^{\infty} f_n x^n+\sum_{n=0}^{\infty}(-1)^n f_n x^n=\sum_{n=0}^{\infty}(1+(-1)^n)f_n x^n.$$

But

$$1+(-1)^n=\begin{cases}2 & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases}$$

so we have

$$f(x)+f(-x)=\sum_{n=0}^{\infty} 2f_{2n}x^{2n}=2\sum_{n=0}^{\infty} f_{2n}(x^2)^n.$$

Let $t=x^2$, then $x=\sqrt{t}$, so

$$\sum_{n=0}^{\infty} f_{2n}t^n=\frac{f\left(\sqrt{t}\right)+f\left(-\sqrt{t}\right)}{2}.$$

In the case of Fibonacci numbers, we have

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2},$$

so

$$\sum_{n=0}^{\infty} F_{2n} t^n = \frac{1}{2}\left(F\left(\sqrt{t}\right) + F\left(-\sqrt{t}\right)\right) = \frac{1}{2}\left(\frac{1}{1-\sqrt{t}-t} + \frac{1}{1+\sqrt{t}-t}\right) = \frac{1+\sqrt{t}-t+1-\sqrt{t}-t}{2(1-\sqrt{t}-t)(1+\sqrt{t}-t)}$$

$$= \frac{2(1-t)}{2\left((1-t)^2 - (\sqrt{t})^2\right)} = \frac{1-t}{1-2t+t^2-t} = \frac{1-t}{1-3t+t^2}.$$

4. *Straightforward solution.* Factoring the denominator $(1-x^2)^2$ we get $(1-x)^2(1+x)^2$, so we need to write our function as the sum of the partial fractions as follows:

$$\frac{1}{(1-x)^2(1+x)^2} = \frac{a_1}{1-x} + \frac{a_2}{(1-x)^2} + \frac{b_1}{1+x} + \frac{b_2}{(1+x)^2}$$

Multiplying both sides by $(1-x)^2$, then letting $x = 1$, we get

$$a_2 = \frac{1}{(1+1)^2} = \frac{1}{4}.$$

Multiplying both sides by $(1+x)^2$, then letting $x = -1$, we get

$$b_2 = \frac{1}{(1-(-1))^2} = \frac{1}{4}.$$

Therefore,

$$\frac{a_1}{1-x} + \frac{b_1}{1+x} = \frac{1}{(1-x)^2(1+x)^2} - \frac{1}{4(1-x)^2} - \frac{1}{4(1+x)^2} = \frac{4-(1-x)^2-(1+x)^2}{4(1-x)^2(1+x)^2}$$

$$= \frac{2-2x^2}{4(1-x)^2(1+x)^2} = \frac{2(1+x)(1-x)}{4(1-x)^2(1+x)^2} = \frac{1}{2(1+x)(1-x)}$$

Thus,

$$\frac{a_1}{1-x} + \frac{b_1}{1+x} = \frac{1}{2(1+x)(1-x)}.$$

Multiplying both sides by $1-x$, then letting $x = 1$, we get

$$a_1 = \frac{1}{2(1+1)} = \frac{1}{4}.$$

Multiplying both sides by $1+x$, then letting $x = -1$, we get

$$b_1 = \frac{1}{2(1-(-1))} = \frac{1}{4}.$$

Thus, $a_1 = a_2 = b_1 = b_2 = 1/4$. Now expanding our partial fractions into series we get

$$\frac{1}{(1-x^2)^2} = \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)(-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{1+(n+1)+(-1)^n(n+1)+(-1)^n}{4} x^n = \sum_{n=0}^{\infty} \frac{(1+(-1)^n)(n+2)}{4} x^n,$$

so the coefficient at x^n is $\frac{n}{2}+1$ if n is even, and 0 if n is odd. Therefore,

$$\frac{1}{(1-x^2)^2} = \sum_{k=0}^{\infty}(k+1)x^{2k}.$$

Whew! Finally.

Sneaky solution. Let $t = x^2$. Then

$$\frac{1}{(1-x^2)^2} = \frac{1}{(1-t)^2} = \sum_{n=0}^{\infty}\binom{n+2-1}{2-1}t^n = \sum_{n=0}^{\infty}(n+1)t^n = \sum_{n=0}^{\infty}(n+1)(x^2)^n = \sum_{n=0}^{\infty}(n+1)x^{2n}.$$

That's all!

5. Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the generating function of $\{g_n\}$. We have $g_0 = 0$, $g_1 = 1$, and $g_n = -g_{n-1} + (g_n) * (g_n)$ for $n \geq 2$ (the second summand is the convolution of the sequence $\{g_n\}$ with itself). Taking the generating functions of both sides, we get

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} g_n x^n = g_0 + g_1 x + \sum_{n=2}^{\infty} g_n x^n = x + \sum_{n=2}^{\infty}(-g_{n-1} + (g_n)*(g_n))x^n \\ &= x + \sum_{n=2}^{\infty}(-g_{n-1})x^n + \sum_{n=2}^{\infty}((g_n)*(g_n))x^n \\ &= x - x\sum_{n=1}^{\infty} g_n x^n + \sum_{n=0}^{\infty}((g_n)*(g_n))x^n - g_0g_0 - (g_0g_1 + g_1g_0)x \\ &= x - x(G(x) - g_0) + G(x)G(x) - 0 - 0x = x - xG(x) + (G(x))^2 \end{aligned}$$

Hence, the functional equation is

$$G(x) = x - xG(x) + (G(x))^2$$

i.e.

$$G(x)^2 - (x+1)G(x) + x = 0 \quad \iff \quad (G(x)-x)(G(x)-1) = 0.$$

Solving this for $G(x)$, we get $G(x) = x$ or $G(x) = 1$. But $g_0 = 0$, $g_1 = 1$, so $G(x) = x$. Therefore, $g_n = 0$ for all $n \geq 2$.

We can also prove this fact by induction on n without using the generating functions. However, that would be somewhat of an ad hoc solution.

6. Considering first several derivatives of e^{e^x} it is easy to guess (and prove by induction) that the general form of the nth derivative should be

$$\frac{d^n}{dx^n}\left(e^{e^x}\right) = e^{e^x}\sum_{k=0}^{n} a(n,k)e^{kx},$$

where $a(n,k)$ are some constants. Again, considering first several derivatives of e^{e^x} (go at least up to the third or fourth derivative), we can guess that $a(n,k) = S(n,k)$, the Stirling number of the second kind. To prove this, we need to show that $a(n,k)$ satisfies the same recurrence relation and initial conditions as $S(n,k)$. We know that $a(0,0) = 1$, $a(n,0) = 0$ if $n > 0$ (both facts are easy to prove since the 0th derivative of e^{e^x} is the function itself, i.e. e^{e^x}). Thus, we only need to show that

$$a(n,k) = a(n-1,k-1) + k\,a(n-1,k) \quad \text{for all } (n,k) \neq (0,0).$$

To prove this, consider let $n > 0$ and consider the nth derivative of e^{e^x},

$$\frac{d^n}{dx^n}\left(e^{e^x}\right) = e^{e^x}\sum_{k=0}^{n} a(n,k)e^{kx}.$$

It is convenient to define $a(n,k)=0$ for $k>n$ and $k<0$ so that we can write

$$\frac{d^n}{dx^n}\left(e^{e^x}\right) = e^{e^x}\sum_{k=-\infty}^{\infty} a(n,k)e^{kx}.$$

We know that the nth is the the derivative of the $(n-1)$-st derivative, i.e.

$$\begin{aligned}
e^{e^x}\sum_{k=-\infty}^{\infty} a(n,k)e^{kx} &= \frac{d}{dx}\left(e^{e^x}\sum_{k=-\infty}^{\infty} a(n-1,k)e^{kx}\right) \\
&= e^x e^{e^x}\sum_{k=-\infty}^{\infty} a(n-1,k)e^{kx} + e^{e^x}\sum_{k=-\infty}^{\infty} a(n-1,k)ke^{kx} \\
&= e^{e^x}\sum_{k=-\infty}^{\infty} a(n-1,k)e^{(k+1)x} + e^{e^x}\sum_{k=-\infty}^{\infty} ka(n-1,k)e^{kx} \\
&= e^{e^x}\left(\sum_{k=-\infty}^{\infty} a(n-1,k-1)e^{kx} + \sum_{k=-\infty}^{\infty} ka(n-1,k)e^{kx}\right) \\
&= e^{e^x}\sum_{k=-\infty}^{\infty} (a(n-1,k-1)+ka(n-1,k))e^{kx}
\end{aligned}$$

Comparing the coefficients at e^{kx} in the first and last expression in the above chain of equalities, we get the desired recurrence relation. This ends the proof.