Math 304 **Homework 4** **Due 11/21/2002**

1. Show that the generating function of the sequence of central binomial coefficients $\binom{2n}{n}$ is $1/\sqrt{1-4x}$, i.e.
$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$
Hint: You might try the same proof we did for the Catalan numbers. However, note that you already know that the generating function of $C_n = \frac{1}{n+1}\binom{2n}{n}$ is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, so there might be a nice shortcut.

2. The *ballot numbers* $b(n,k)$ are defined as
$$b(n,k) = \frac{k}{2n+k}\binom{2n+k}{n}, \quad k \geq 1.$$
   (a) Show that $C_n = b(n,1)$, and
$$b(n,k) = \frac{k}{n+k}\binom{2n+k-1}{n}, \quad b(n,k) = \binom{2n+k-1}{n} - \binom{2n+k-1}{n-1}.$$
   (b) Consider an election with 2 candidates, D and R, where D got n votes and R got $n+k-1$ votes, $k \geq 1$. After each vote was cast (for R or for D), we calculated by how many votes R was currently ahead of D, and it turned out that R was never ahead of D by k votes (i.e. was behind D, tied with D, or ahead of D by at most $k-1$ votes). Prove that the number of possible vote sequences which satisfy this condition is $b(n,k)$. *Hint:* Use the second formula in part (a) and a combinatorial proof similar to that of $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

3. Let F_n be the n-th Fibonacci number. Find the generating function for the sequence of *even* Fibonacci numbers, i.e.
$$\sum_{n=0}^{\infty} F_{2n} t^n.$$
Hint: Let $F(x) = \sum_{n=0}^{\infty} F_n x^n$. Consider the coefficients of $F(x) + F(-x)$. Let $t = x^2$ and express $F(x) + F(-x)$ as a rational function of t (no x or $\sqrt{t}$) in closed form.

4. Let $\{g_n\}_{n=0}^{\infty}$ be an integer sequence given by
$$g_n = -g_{n-1} + \sum_{i=0}^{n} g_i g_{n-i} \;\text{ for } n \geq 2; \quad g_0 = 0,\; g_1 = 1.$$
Find g_n for any integer $n \geq 0$. *Hint:* One possible approach is to first determine the generating function of $\{g_n\}$.

5. Find the coefficient of x^n in the power series of

$$f(x) = \frac{1}{(1-x^2)^2},$$

first by the method of partial fractions, and second, give a much simpler derivation by using a substitution.

6. We want to find a formula for the nth derivative of $e^{e^x} = \exp(\exp(x))$. Differentiate it a few times, study the pattern, and conjecture the form of the answer in general, including some constants to be determined. Then find a recurrence formula for the constants in question, and identify them as some "famous" numbers that we have studied.